Review of Fundamental Equations

Supplementary notes on Section 1.2 and 1.3

- Introduction of the velocity potential:

  irrotational motion: $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 0$

  identity in the vector analysis: $\nabla \times \nabla\phi \equiv 0$

  $\longrightarrow\ \ \boldsymbol{u} = \nabla\phi$

- Basic conservation principles:

  (1) Conservation of mass

  $\longrightarrow$ Continuity equation $\quad \nabla \cdot \boldsymbol{u} = 0$

  (2) Conservation of momentum

  $\longrightarrow$ Euler's equation (for inviscid fluid) $\quad \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\dfrac{1}{\rho}\nabla p + \boldsymbol{K} \quad (\,\boldsymbol{K} = g\boldsymbol{k}\,)$

For ideal fluid

From (1) : Laplace's equation $\quad \nabla\cdot\nabla\phi = \nabla^2\phi = 0$

From (2) : Bernoulli's equation $\quad -\dfrac{1}{\rho}(p - p_a) = \dfrac{\partial\phi}{\partial t} + \dfrac{1}{2}\nabla\phi\cdot\nabla\phi - gz$

because of a relation

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u})$$

Euler's equation becomes $\quad \nabla\left[\dfrac{\partial\phi}{\partial t} + \dfrac{1}{2}\nabla\phi\cdot\nabla\phi + \dfrac{p}{\rho} - gz\right] = 0$

Annex

The other identity in the vector analysis: $\quad \nabla\cdot(\nabla\times\boldsymbol{A}) \equiv 0$

From the continuity equation $\nabla\cdot\boldsymbol{u} = 0$

$\longrightarrow \quad \boldsymbol{u} = \nabla\times\boldsymbol{A} \qquad$ ($\boldsymbol{A}$: defined as the vector potential)

For 2-D flows $\boldsymbol{u} = (u, v, 0)$ and thus

the vector potential must be $\boldsymbol{A} = (0, 0, \psi)$

$\longrightarrow \quad u = \dfrac{\partial\psi}{\partial y}, \quad v = -\dfrac{\partial\psi}{\partial x}$

which is known as the stream function for 2-D flows.

Boundary Conditions

(1) Kinematic condition

Fluid particles on a wetted boundary surface, described by $F(x, y, z, t) = 0$, always follow the movement of the boundary surface. Namely, even after a short time interval, the fluid particles remain on the boundary surface. Thus we can write as $F(x + u\Delta t, y + v\Delta t, z + w\Delta t, t + \Delta t) = 0$. Then considering subtraction of these two and applying a Taylor-series expansion with respect to Δt, we may have the following result:

$$\begin{aligned} &F(x + u\Delta t,\ y + v\Delta t,\ z + w\Delta t,\ t + \Delta t) - F(x, y, z, t) \\ &\qquad = u\Delta t\,\frac{\partial F}{\partial x} + v\Delta t\,\frac{\partial F}{\partial y} + w\Delta t\,\frac{\partial F}{\partial z} + \Delta t\,\frac{\partial F}{\partial t} + O\big[(\Delta t)^2\big] = 0 \end{aligned} \tag{1}$$

Then dividing the above by Δt and taking the limit of $\Delta \to 0$, we have the following result:

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} \equiv \frac{DF}{Dt} = 0 \tag{2}$$

With the definition of the velocity potential $\boldsymbol{u} = (u, v, w) = \nabla\phi$, this result can be written as

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla\phi \cdot \nabla F = 0 \qquad \text{on } F = 0 \tag{3}$$

Dividing both sides with $|\nabla F|$ and noting the definition of the normal vector $\boldsymbol{n} = \nabla F/|\nabla F|$, we may have

$$\nabla\phi \cdot \boldsymbol{n} = \frac{\partial \phi}{\partial n} = -\frac{1}{|\nabla F|}\frac{\partial F}{\partial t} \;\left(\equiv V_n \right) \tag{4}$$

(2) Free-surface condition

Provided that the elevation of free surface is expressed as $z = \zeta(x, y, t)$, the kinematic boundary condition is given as follows:

$$F(x, y, z, t) = z - \zeta(x, y, t) = 0 \tag{5}$$

$$\frac{DF}{Dt} = -\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = \zeta(x, y, t) \tag{6}$$

Here we note that both ϕ and ζ are unknown. Thus we need one more boundary condition on the free surface; that is, the dynamic boundary condition which states that the pressure on the free surface is equal to the atmospheric pressure:

$$-\frac{1}{\rho}\left(p - p_a\right) = \frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi - g\zeta = 0 \quad \text{on } z = \zeta(x, y, t) \tag{7}$$

<u>Linearization</u>

Assuming that both ϕ and ζ are of small quantities and retaining only the first-order terms in ϕ and ζ, we may have the followings:

$$-\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial z} + O(\zeta\phi) = 0 \tag{8}$$

$$\zeta = \frac{1}{g}\frac{\partial \phi}{\partial t} + O(\phi^2) = 0 \tag{9}$$

Eliminating ζ from (6) and (7), it follows that

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \phi}{\partial z} + O(\phi^2, \zeta\phi) = 0 \quad \text{on } z = \zeta(x, y, t) \tag{10}$$

Furthermore applying the Taylor-series expansion around the undisturbed free surface ($z = 0$):

$$\phi(x, y, z, t) = \phi(x, y, 0, t) + \zeta \left(\frac{\partial \phi}{\partial z} \right)_{z=0} + \cdots \tag{11}$$

and neglecting higher-order terms resulting from this Taylor expansion, the final result takes the following form:

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = 0 \tag{12}$$

Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Section 1.5

Plane Progressive Waves

The free-surface elevation:

$$\zeta = \frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} = A\cos(\omega t - kx) = \mathrm{Re}\big[\,A\,e^{-ikx}\,e^{i\omega t}\,\big] \tag{1}$$

The phase function $\omega t - kx$, which represents a wave propagating in the positive x-axis, because $f(\omega t - kx)$ satisfies

$$\left.\begin{aligned} \frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = \big(\,\omega - ck\,\big)f' = 0 \\ c = \frac{\omega}{k} > 0 \end{aligned}\right\} \tag{2}$$

Here k is the *wavenumber*, ω is the *circular* (or angular) *frequency* , and c is the *phase velocity.*

Dispersion Relation

The general solution to be obtained from Laplace's equation is assumed to be in a form

$$\phi(x,z,t) = \mathrm{Re}\big[\,Z(z)\,e^{-ikx}\,e^{i\omega t}\,\big] \tag{3}$$

Here it should be noted that $Z(z)$ can be complex. Then a general solution for $Z(z)$ is given by

$$Z(z) = C\,e^{kz} + D\,e^{-kz} \tag{4}$$

where C and D are unknown to be determined from boundary conditions.

The free-surface and bottom boundary conditions for $Z(z)$ are written as follows:

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \phi}{\partial z} = 0 \;\longrightarrow\; -\omega^2 Z - g\frac{dZ}{dz} = 0 \quad \text{on } z = 0 \tag{5}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad\qquad \longrightarrow\; \frac{dZ}{dz} = 0 \quad \text{on } z = h \tag{6}$$

Substituting (4) into (5) and (6) gives the following:

$$\left.\begin{aligned} C(\omega^2 + gk) + D(\omega^2 - gk) = 0 \\ C\,e^{kh} - D\,e^{-kh} = 0 \end{aligned}\right\} \tag{7}$$

We note that both unknowns, C and D, cannot be determined uniquely only from these equations (because both equations above are homogeneous ones). However, in order to have non-trivial solutions, the following relation must hold:

$$\begin{vmatrix} \omega^2 + gk & \omega^2 - gk \\ e^{kh} & -\,e^{-kh} \end{vmatrix} = 0 \tag{8}$$

$$\longrightarrow \qquad -\,e^{-kh}(\omega^2 + gk) -\, e^{kh}(\omega^2 - gk) = 0$$

$$\longrightarrow \qquad gk\big(\,e^{kh} -\, e^{-kh}\big) - \omega^2\big(\,e^{kh} +\, e^{-kh}\big) = 0$$

$$\longrightarrow \qquad k\tanh kh = \frac{\omega^2}{g} \tag{9}$$

This is the *dispersion relation*, implying that the wavenumber (wavelength) and the frequency (period) are mutually dependent parameters.

It should be noted that we can eliminate just one unknown from (7) and the resultant equation can be written in the form

$$Z(z) = \tilde{C}\,\frac{\cosh k(z-h)}{\cosh kh} \qquad \tilde{C} \equiv 2C\,e^{kh}\cosh kh \tag{10}$$

Mathematically speaking, (9) is the eigen-value equation (the equation for eigen values) and (10) is the associated eigen solution or homogeneous solution.

In order to determine the remaining unknown coefficient in (10), we must specify the free-surface elevation given by (1). From (3) and (10), we have

$$\phi(x,z,t) = \mathrm{Re}\Big[\,\tilde{C}\,\frac{\cosh k(z-h)}{\cosh kh}\,e^{-ikx}\,e^{i\omega t}\,\Big] \tag{11}$$

$$\longrightarrow \quad \zeta = \frac{1}{g}\frac{\partial\phi}{\partial t}\bigg|_{z=0} = \mathrm{Re}\Big[\,\frac{i\omega}{g}\tilde{C}\,e^{-ikx}\,e^{i\omega t}\,\Big] \tag{12}$$

By comaring the above with (1), we can determine $\tilde{C}$ as follows:

$$\frac{i\omega}{g}\,\tilde{C} = A \quad \longrightarrow \quad \tilde{C} = \frac{gA}{i\omega} \tag{13}$$

Then the solution can be obtained in the form:

$$\phi = \mathrm{Re}\Big[\,\frac{gA}{i\omega}\frac{\cosh k(z-h)}{\cosh kh}\,e^{-ikx}\,e^{i\omega t}\,\Big] = \frac{gA}{\omega}\frac{\cosh k(z-h)}{\cosh kh}\sin(\omega t - kx) \tag{14}$$

Approximation of $\tanh kh \simeq 1$ is valid for $kh > \pi$ with error less than 0.4 %. This means that if

$$kh = \frac{2\pi h}{\lambda} > \pi \quad \longrightarrow \quad h > \frac{\lambda}{2} \tag{15}$$

is satisfied, the dispersion relation can be practically the same as that for deep water.

For the deep-water case, several relations become rather simple as follows:

$$k = K = \frac{\omega^2}{g} = \frac{2\pi}{\lambda}, \quad T = \frac{2\pi}{\omega} = \sqrt{\frac{2\pi\lambda}{g}} \simeq 0.8\sqrt{\lambda} \quad (\,\lambda \simeq 1.56T^2\,) \tag{16}$$

$$\begin{aligned} \phi &= \frac{gA}{\omega}\,e^{-kz}\sin(\omega t - kx) \\ &= \mathrm{Re}\Big[\,\frac{gA}{i\omega}\,e^{-kz-ikx}\,e^{i\omega t}\,\Big] \equiv \mathrm{Re}\big[\,\varphi(x,z)\,e^{i\omega t}\,\big] \end{aligned} \tag{17}$$

where

$$\varphi(x,z) = \frac{gA}{i\omega}\,e^{-kz-ikx} \tag{18}$$

Amplitude Dispersion Relation in Deep Water

According to the textbook, the following expression for the phase velocity is obtained:

$$c = \sqrt{\frac{g}{k}}\Big(\,1 + \frac{1}{2}k^2A^2\,\Big) = c^{(1)}\Big(\,1 + \frac{1}{2}k^2A^2\,\Big) \tag{19}$$

If we require $\frac{1}{2}(kA)^2 < 0.005$, a linear wave may be guaranteed and this requirement gives the following estimation:

$$kA < \sqrt{0.01} \quad \longrightarrow \quad \frac{2A}{\lambda} = \frac{H}{\lambda} < \frac{\sqrt{0.01}}{\pi} \simeq \frac{1}{30} \tag{20}$$

Here H/λ is referred to as the wave steepness.

Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Section 1.5.4 and 1.6

Real Part of Eq. (1.65)

We assume that the difference in amplitude is also small, represented as $A_2 - A_1 = \delta A$. Substituting this into Eq. (1.65), we can transform as follows:

$$
\begin{aligned}
&A_1\Big[\,1+\frac{A_2}{A_1}\,e^{i(\delta\omega\cdot t-\delta k\cdot x)}\,\Big] \\
&\quad = A_1\big\{\,1+\,e^{i(\delta\omega\cdot t-\delta k\cdot x)}\big\} + \delta A\,e^{i(\delta\omega\cdot t-\delta k\cdot x)} \\
&\quad = A_1 2\,e^{i\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)}\frac{1}{2}\Big\{\,e^{i\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)}\,+\,e^{-i\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)}\Big\} + \delta A\,e^{i(\delta\omega\cdot t-\delta k\cdot x)} \\
&\quad = 2A_1\cos\Big\{\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)\Big\}\,e^{i\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)} + \delta A\,e^{i(\delta\omega\cdot t-\delta k\cdot x)}
\end{aligned}
\tag{1}
$$

Therefore, taking the real part of Eq. (1.65), we can obtain the following result:

$$
\begin{aligned}
\eta &= \mathrm{Re}\Big\{\,A_1\Big[\,1+\frac{A_2}{A_1}\,e^{i(\delta\omega\cdot t-\delta k\cdot x)}\,\Big]e^{i(\omega_1 t-k_1 x)}\,\Big\} \\
&= 2A_1\cos\Big\{\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)\Big\}\cos\Big\{\Big(\omega_1+\frac{1}{2}\delta\omega\Big)t-\Big(k_1+\frac{1}{2}\delta k\Big)x\Big\} \\
&\qquad +\delta A\cos\big\{(\omega_1+\delta\omega)t-(k_1+\delta k)x\,\big\}
\end{aligned}
\tag{2}
$$

Retaining only the leading term of the above equation gives the following approximation:

$$
\eta = 2A_1\cos\Big\{\frac{1}{2}(\delta\omega\cdot t-\delta k\cdot x)\Big\}\cos\big(\omega_1 t-k_1 x\big)+O(\delta\omega,\delta k,\delta A) \tag{3}
$$

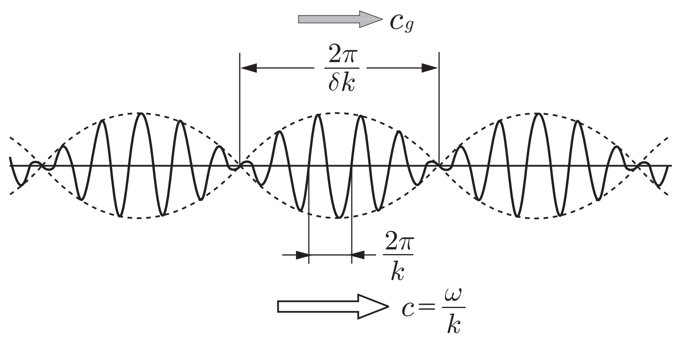

Fig. 1 The amplitude modulation part is an envelope of fundamental carrier waves, and its velocity (group velocity) is given by $\delta\omega/\delta k$.

Calculation of Group Velocity: Eq. (1.69)

The dispersion relation for finite-water depth takes the form

$$
\omega^2 = gk\tanh kh \tag{4}
$$

The definition of the group velocity is given by

$$
c_g = \frac{d\omega}{dk} \tag{5}
$$

Taking first the logarithm of both sides of (4) and then differentiating with respect to k, we may have the following

$$
2\log\omega = \log gk + \log\tanh kh
$$

$$\longrightarrow\ 2\frac{\omega'}{\omega}=\frac{1}{k}+\frac{1}{\tanh kh}\frac{h}{\cosh^2 kh}$$

$$\longrightarrow\ \omega'=\frac{d\omega}{dk}=\frac{1}{2}\frac{\omega}{k}\left\{1+\frac{kh}{\cosh kh\sinh kh}\right\}$$

$$\longrightarrow\ c_g=\frac{1}{2}c\left\{1+\frac{2kh}{\sinh 2kh}\right\} \tag{6}$$

<u>Calculation related to Eq. (1.76)</u>

From Eq. (1.58), we can obtain the followings:

$$\phi(x,y)=\frac{ga}{i\omega}\frac{\cosh k(y-h)}{\cosh kh}\,e^{-ikx} \tag{7}$$

$$\frac{\partial\phi}{\partial x}=-\frac{ga}{\omega}k\,\frac{\cosh k(y-h)}{\cosh kh}\,e^{-ikx} \tag{8}$$

$$=-a\omega\,\frac{\cosh k(y-h)}{\sinh kh}\,e^{-ikx} \qquad \longleftarrow \quad \frac{k}{\cosh kh}=\frac{\omega^2}{g\sinh kh} \tag{9}$$

$$\frac{\partial\phi}{\partial y}=\frac{ga}{i\omega}k\,\frac{\sinh k(y-h)}{\cosh kh}\,e^{-ikx}=-ia\omega\,\frac{\sinh k(y-h)}{\sinh kh}\,e^{-ikx} \tag{10}$$

Therefore it follows that

$$\left|\frac{\partial\phi}{\partial x}\right|^2+\left|\frac{\partial\phi}{\partial y}\right|^2=(a\omega)^2\,\frac{\cosh^2 k(y-h)+\sinh^2 k(y-h)}{\sinh^2 kh}=(a\omega)^2\,\frac{\cosh 2k(y-h)}{\sinh^2 kh} \tag{11}$$

$$\int_0^h\frac{\cosh 2k(y-h)}{\sinh^2 kh}\,dy=\left[\frac{\sinh 2k(y-h)}{2k\sinh^2 kh}\right]_0^h$$

$$=\frac{\sinh 2kh}{2k\sinh^2 kh}=\frac{\cosh kh}{k\sinh kh}=\frac{g}{\omega^2} \qquad \longleftarrow \quad \frac{1}{k\tanh kh}=\frac{g}{\omega^2} \tag{12}$$

Summarizing these results, we can obtain the following result:

$$\frac{1}{4}\rho\int_0^h\left\{\left|\frac{\partial\phi}{\partial x}\right|^2+\left|\frac{\partial\phi}{\partial y}\right|^2\right\}dy=\frac{1}{4}\rho(a\omega)^2\frac{g}{\omega^2}=\frac{1}{4}\rho g a^2 \tag{13}$$

<u>Calculation related to Eq. (1.78)</u>

From Eq. (7) and Eq. (8), we have

$$i\omega\phi\frac{\partial\phi^*}{\partial x}=-(ga)^2\frac{k}{\omega}\,\frac{\cosh^2 k(y-h)}{\cosh^2 kh}=-(ga)^2\frac{k}{\omega}\,\frac{1}{\cosh^2 kh}\,\frac{1+\cosh 2k(y-h)}{2} \tag{14}$$

$$\int_0^h\frac{1+\cosh 2k(y-h)}{2}\,dy=\frac{1}{2}\left[y+\frac{\sinh 2k(y-h)}{2k}\right]_0^h=\frac{1}{2}\left(h+\frac{\sinh 2kh}{2k}\right)$$

$$=\frac{\sinh 2kh}{4k}\left\{1+\frac{2kh}{\sinh 2kh}\right\}=\frac{\sinh kh\cosh kh}{2k}\left\{1+\frac{2kh}{\sinh 2kh}\right\} \tag{15}$$

Therefore, with the dispersion relation $\tanh kh=\omega^2/gk$, we have the following result:

$$\frac{1}{2}\rho\,\mathrm{Re}\int_0^h(i\omega\phi)\frac{\partial\phi^*}{\partial x}\,dy=-\frac{1}{4}\rho(ga)^2\frac{1}{\omega}\frac{\sinh kh}{\cosh kh}\left\{1+\frac{2kh}{\sinh 2kh}\right\}$$

$$=-\frac{1}{4}\rho(ga)^2\frac{1}{\omega}\frac{\omega^2}{gk}\left\{1+\frac{2kh}{\sinh 2kh}\right\}=-\frac{1}{4}\rho g a^2\frac{\omega}{k}\left\{1+\frac{2kh}{\sinh 2kh}\right\} \tag{16}$$

Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Section 2.1

Meaning of Eq. (2.2)

Let us consider the total amount of flux Q across the boundary C of the fluid (in the 2D problem) denoted as S.

$$Q = \int_C \frac{\partial \phi}{\partial n}\, d\ell = \int_C \boldsymbol{n} \cdot \nabla\phi\, d\ell = \iint_S \nabla \cdot \nabla\phi\, dS = \iint_S \nabla^2 \phi\, dS$$

If there is no singularity (like source), $Q = 0$ due to the conservation of mass. However, if there is a source within S, the flux Q must be not zero but equal to the amount coming from the point of source; which is assumed to be unit in the present case. Thus when the source is located at $(x, y) = (0, \eta)$,

$$Q = \iint_S \nabla^2 \phi\, dS = 1 \quad \longrightarrow \quad \nabla^2 \phi = \delta(x)\, \delta(y - \eta)$$

Because, by the definition of the delta function, it follows that

$$\int_{-\epsilon}^{+\epsilon} \delta(x)\, dx = 1, \quad \int_{\eta-\epsilon}^{\eta+\epsilon} \delta(y - \eta)\, dy = 1$$

Solution Methods for 2-D Laplace Equation

1. Solution in the Cartesian Coordinate System

The x-axis is taken in the horizontal direction and the positive y-axis is taken vertically downward, with the origin on the undisturbed free surface.

$$\nabla^2 \varPhi = \frac{\partial^2 \varPhi}{\partial x^2} + \frac{\partial^2 \varPhi}{\partial y^2} = 0 \tag{17}$$

Let us apply the method of separation of variables; that is, the solution is assumed to be given in a form of $\varPhi = X(x)Y(y)$. Substituting this into (17), we have

$$X'' Y + X Y'' = 0 \tag{18}$$

Therefore

$$\frac{X''}{X} = -\frac{Y''}{Y} \equiv -k^2 \text{ or } +k^2 \tag{19}$$

$$\longrightarrow \quad \left.\begin{aligned} X'' \pm k^2 X = 0 \\ Y'' \mp k^2 Y = 0 \end{aligned}\right\} \tag{20}$$

Constant k is arbitrary, not necessarily integer. If we require that the solution must be finite at infinity ($y \to \infty$ or $|x| \to \infty$), fundamental solutions of (20) are given as

$$-k^2 \longrightarrow e^{-ky}\Big\{ A\cos kx + B\sin kx \Big\} \tag{21}$$

$$+k^2 \longrightarrow e^{-k|x|}\Big\{ A\cos ky + B\sin ky \Big\} \tag{22}$$

Here A and B are constants, possibly functions of parameter k, and thus a general solution may be written in the form

$$-k^2 \longrightarrow \varPhi = \int_0^\infty e^{-ky}\Big\{ A(k)\cos kx + B(k)\sin kx \Big\}\, dk \tag{23}$$

$$+k^2 \longrightarrow \varPhi = \int_0^\infty e^{-k|x|}\Big\{ A(k)\cos ky + B(k)\sin ky \Big\}\, dk \tag{24}$$

2. Solution in the Polar Coordinate System

$$\nabla^2 \varPhi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varPhi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varPhi}{\partial \theta^2} = 0 \tag{25}$$

Applying the method of separation of variables, the solution can be assumed to be of the form $\varPhi = R(r)\varTheta(\theta)$. Substituting this into (25), we have the following:

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\frac{1}{R} = -\frac{d^2\varTheta}{d\theta^2}\frac{1}{\varTheta} \equiv n^2 \tag{26}$$

Therefore

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 R = 0 \tag{27}$$

$$\frac{d^2\varTheta}{d\theta^2} + n^2\varTheta = 0 \tag{28}$$

Here we should note that the parameter n introduced in (26) must be integer, because fundamental solutions with respect to θ to be given by (28) must be periodic (the values for $\theta = 0$ and $\theta = 2\pi$ must be the same). This situation is different from the case considered in the Cartesian coordinate system.

The elementary solutions for (27) and (28) are given by

$$R = \{\, r^n \,,\ r^{-n} \,\} \,, \qquad \Theta = \{\, \cos n\theta \,,\ \sin n\theta \,\} \tag{29}$$

If we require regularity of a solution at infinity ($r \to \infty$), a general solution can be written as

$$\Phi = \sum_{n=1}^{\infty} \frac{1}{r^n} \Big\{\, A_n \cos n\theta + B_n \sin n\theta \,\Big\} \tag{30}$$

For the case of $n = 0$, the solution will be independent of θ. (The possibility of $\Theta = \theta$ which may be obtained from (28) should be excluded from the condition of periodicity.) In this case,

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) = 0 \;\longrightarrow\; r \frac{dR}{dr} = C \;\longrightarrow\; R = C \log r \tag{31}$$

Thus we have the following equation as a general solution:

$$\Phi = C \log r + \sum_{n=1}^{\infty} \frac{1}{r^n} \Big\{\, A_n \cos n\theta + B_n \sin n\theta \,\Big\} \tag{32}$$

3. Relations of Solutions

There must be some relations between (23) and (32), because both are solutions for the same 2-D Laplace equation. In order to see those relations, let us consider the following integral:

$$\begin{aligned} I &= \int_0^\infty k^{n-1}\, e^{-ky} \big(\, \cos kx + i \sin kx \,\big)\, dk \\ &= \int_0^\infty k^{n-1}\, e^{-ky+ikx}\, dk = \int_0^\infty k^{n-1}\, e^{-k(y-ix)}\, dk \end{aligned} \tag{33}$$

Introducing the relations $y = r\cos\theta$, $x = r\sin\theta$, we can perform the integral with respect to k as follows:

$$\begin{aligned} I &= \int_0^\infty k^{n-1}\, e^{-kr\exp(-i\theta)}\, dk = \frac{e^{i\theta}}{r}(n-1) \int_0^\infty k^{n-2}\, e^{-kr\exp(-i\theta)}\, dk \\ &= \left(\frac{e^{i\theta}}{r} \right)^{n-1} (n-1)! \int_0^\infty e^{-kr\exp(-i\theta)}\, dk = \left(\frac{e^{i\theta}}{r} \right)^{n} (n-1)! \end{aligned} \tag{34}$$

Therefore, separating the real and imaginary parts, we have

$$\left. \begin{aligned} \int_0^\infty k^{n-1}\, e^{-ky} \cos kx \, dk = (n-1)!\, \frac{\cos n\theta}{r^n} \\ \int_0^\infty k^{n-1}\, e^{-ky} \sin kx \, dk = (n-1)!\, \frac{\sin n\theta}{r^2} \end{aligned} \right\} \tag{35}$$

As the next, we will consider the following integral as the case of $n = 0$

$$J = \int_0^\infty \frac{1}{k}\, e^{-ky} \cos kx \, dk \tag{36}$$

Then, differentiating with respect to y, we have

$$\frac{dJ}{dy} = -\int_0^\infty e^{-ky} \cos kx \, dk = -\frac{y}{x^2+y^2}$$

$$\begin{aligned} J &= -\int_0^y \frac{\eta}{x^2+\eta^2}\, d\eta + C \\ &= -\frac{1}{2}\log(x^2+y^2) + C(x) = -\log r + C \end{aligned}$$

Namely

$$\int_0^\infty \frac{1}{k}\, e^{-ky} \cos kx\, dk = -\log r + C \tag{37}$$

Equations (35) and (37) are desired relations between the solutions in the Cartesian and polar coordinate systems.

4. Application of Fourier Transform

One-dimensional Fourier transform is defined in the form:

$$\left.\begin{aligned} f^*(k,y) &= \int_{-\infty}^{\infty} f(x,y)\, e^{-ikx}\, dx \\ f(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k,y)\, e^{ikx}\, dk \end{aligned}\right\} \tag{38}$$

Similarly two-dimensional Fourier transform can be expressed as

$$\left.\begin{aligned} f^{**}(k,\ell) &= \iint_{-\infty}^{\infty} f(x,y)\, e^{-ikx-i\ell y}\, dxdy \\ f(x,y) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} f^{**}(k,\ell)\, e^{ikx+i\ell y}\, dkd\ell \end{aligned}\right\} \tag{39}$$

Note the following relations in the Fourier transform:

$$\int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = 1 \tag{40}$$

$$\int_{-\infty}^{\infty} \frac{df(x)}{dx}\, e^{-ikx}\, dx = \Big[\, f(x)\, e^{-ikx}\,\Big]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = ikf^*(k) \tag{41}$$

Let us consider a fundamental solution, corresponding to a hydrodynamic source with unit strength situated at $(x, y) = (0, \eta)$, which satisfies

$$\nabla^2\phi = \delta(x)\delta(y-\eta) \tag{42}$$

Applying the Fourier transform with respect to x to (42) and noting (40) and (41), we have

$$\frac{d^2\phi^*}{dy^2} - k^2\phi^* = \delta(y-\eta) \tag{43}$$

A general solution of (43), valid except at $y = \eta$, is given as

$$\phi^*(k,y) = A\, e^{|k|y} + B\, e^{-|k|y} \tag{44}$$

Unknown coefficients, A and B, must be determined from boundary conditions and by taking account of the singularity at $y = \eta$. First, to avoid the singularity at $y = \eta$ for a moment, we divide the region into the lower (denoted as Region 1) and upper (Region 2) parts, separated at point $y = \eta$. Since the region is considered unbounded (no outer boundaries), the solution must be zero at both infinities ($y \to \pm\infty$). Thus the solutions in the lower (written as ϕ_1^*) and upper (ϕ_2^*) regions are given as

$$\phi_1^* = A\, e^{|k|y}\,, \quad \phi_2^* = B\, e^{-|k|y} \tag{45}$$

(Here the positive y-axis is taken vertically upward.)

To take account of the singularity at $y = \eta$, let us integrate (42) over a small region crossing $y = \eta$; i.e. $\eta - \epsilon < y < \eta + \epsilon$ (where ϵ is assumed to very small). The result takes a form

$$\left[\,\frac{d\phi^*}{dy}\,\right]_{\eta-\epsilon}^{\eta+\epsilon} - k^2\int_{\eta-\epsilon}^{\eta+\epsilon} \phi^*\, dy = \int_{\eta-\epsilon}^{\eta+\epsilon} \delta(y-\eta)\, dy = 1 \tag{46}$$

This relation can be satisfied, provided that

$$\phi_2^* = \phi_1^* \,, \quad \frac{d\phi_2^*}{dy} - \frac{d\phi_1^*}{dy} = 1 \quad \text{at } y = \eta \tag{47}$$

Substituting (45) in (47) gives the followings:

$$\left.\begin{aligned} B\, e^{-|k|\eta} - A\, e^{|k|\eta} = 0 \\ -|k|\big(\, B\, e^{-|k|\eta} + A\, e^{|k|\eta} \,\big) = 1 \end{aligned}\right\} \tag{48}$$

From these we can determine A and B as follows:

$$A = -\frac{1}{2|k|}\, e^{-|k|\eta} \,, \quad B = -\frac{1}{2|k|}\, e^{|k|\eta} \tag{49}$$

Substituting these into (45) gives the desired solution in the form

$$\left.\begin{aligned} \phi_1^* &= -\frac{1}{2|k|}\, e^{|k|(y-\eta)} \quad && \text{for } y - \eta < 0 \\ \phi_2^* &= -\frac{1}{2|k|}\, e^{-|k|(y-\eta)} \quad && \text{for } y - \eta > 0 \end{aligned}\right\} \tag{50}$$

These two can be written as a unified form

$$\phi^*(k, y) = -\frac{1}{2|k|}\, e^{-|k||y-\eta|} \tag{51}$$

As the next step for obtaining the solution in the physical domain, the inverse Fourier transform must be considered, which becomes

$$\begin{aligned} \phi(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ -\frac{1}{2|k|}\, e^{-|k||y-\eta|} \right\} e^{ikx}\, dk = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{|k|}\, e^{-|k||y-\eta|+ikx}\, dk \\ &= -\frac{1}{2\pi} \int_0^{\infty} \frac{1}{k}\, e^{-k|y-\eta|} \cos kx\, dk = -\frac{1}{2\pi}\, \mathrm{Re} \int_0^{\infty} \frac{1}{k}\, e^{ik(x+i|y-\eta|)}\, dk \end{aligned} \tag{52}$$

In relation to this integral, let us consider the following integral:

$$F(r) = \int_0^{\infty} \frac{1}{k}\, e^{ikr}\, dk \tag{53}$$

This integral can be evaluated as follows:

$$\begin{aligned} \frac{dF}{dr} &= i \int_0^{\infty} e^{ikr}\, dk = \left[\, \frac{1}{r}\, e^{ikr} \,\right]_0^{\infty} = -\frac{1}{r} \\ &\longrightarrow \quad F(r) = -\log r + C \end{aligned} \tag{54}$$

Therefore we can recast the result of (52) in the form

$$\phi(x, y) = -\frac{1}{2\pi}\, \mathrm{Re}\Big[-\log\big(x + i|y-\eta|\big) \Big] = \frac{1}{2\pi} \log r \,, \quad r = \sqrt{x^2 + (y-\eta)^2} \tag{55}$$

The procedure explained above looks complicated, but it will be useful in understanding derivation of the free-surface Green function.

To see another method using the Fourier transform, let us apply the two-dimensional Fourier transform to the following Laplace equation:

$$\nabla^2 \phi = \delta(x)\delta(y) \tag{56}$$

With relations of (40) and (41), it follows that

$$-\big(\, k^2 + \ell^2 \,\big)\phi^{**} = 1 \quad \longrightarrow \quad \phi^{**}(k, \ell) = -\frac{1}{k^2 + \ell^2} \tag{57}$$

Thus the inverse Fourier transform provides the following expression:

$$\phi(x,y) = -\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{1}{k^2+\ell^2}\, e^{ikx+i\ell y}\, dk d\ell \tag{58}$$

Let us first perform the integral with respect to ℓ:

$$I(k) = \int_{-\infty}^{\infty} \frac{e^{i\ell y}}{\ell^2+k^2}\, d\ell \tag{59}$$

Since there are singular points at $\ell = \pm i|k|$, a contour integral taken in the upper half complex plane (for the case of $y > 0$) gives the following result:

$$I(k) = 2\pi i \frac{e^{-|k|y}}{2i|k|} = \frac{\pi}{|k|}\, e^{-|k|y} \quad \text{for } y > 0 \tag{60}$$

Substituting this in (58), we have the result:

$$\phi(x,y) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{|k|}\, e^{-|k||y|+ikx}\, dk = -\frac{1}{2\pi} \int_{0}^{\infty} \frac{1}{k}\, e^{-k|y|} \cos kx\, dk \tag{61}$$

This is the same as (52) and thus the final expression will be the same as (55), as expected.

Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Sections 3.1 and 3.2

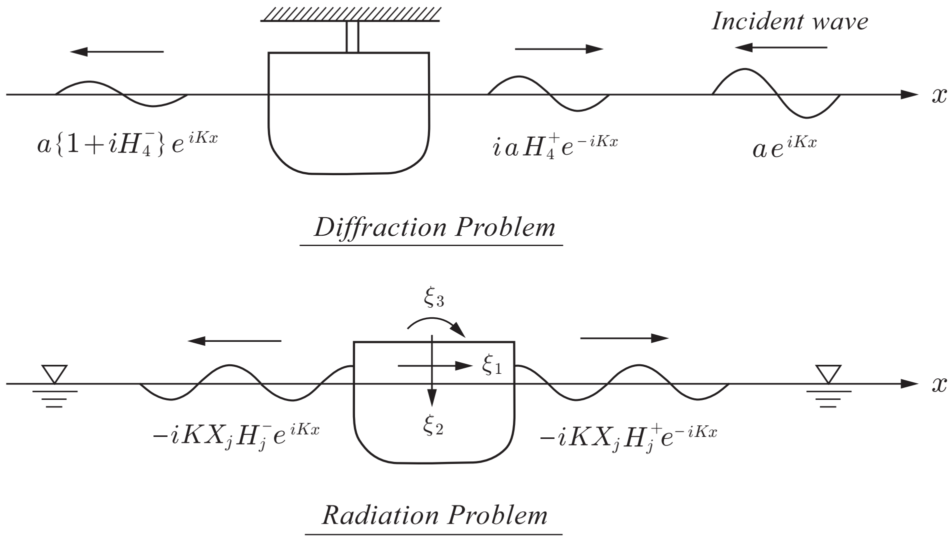

Fig.1 Schematic illustration for the diffraction and radiation problems.

Decomposition of the velocity potential

$$\phi(\boldsymbol{x}) = \frac{ga}{i\omega}\Big\{\varphi_0(\boldsymbol{x}) + \varphi_4(\boldsymbol{x})\Big\} + \sum_{j=1}^{3} i\omega X_j\,\varphi_j(\boldsymbol{x}) \tag{1}$$

$$\varphi_0(\boldsymbol{x}) = e^{-Ky+iKx}\,; \quad \text{which is given as input} \tag{2}$$

Body boundary condition

$$\frac{\partial\phi}{\partial n} = \sum_{j=1}^{3} i\omega X_j\,n_j \qquad (\,n_3 \equiv xn_2 - yn_1\,) \tag{3}$$

$$\longrightarrow \quad \frac{\partial}{\partial n}\big(\varphi_0 + \varphi_4\big) = 0\,, \quad \frac{\partial\varphi_j}{\partial n} = n_j \quad (j = 1,2,3) \tag{4}$$

Body-disturbance waves

$$\zeta(x) = \frac{i\omega}{g}\,\phi(x,0) = a\Big\{\varphi_0(x,0) + \varphi_4(x,0) - K\sum_{j=1}^{3}\frac{X_j}{a}\,\varphi_j(x,0)\Big\} \tag{5}$$

$$\varphi_j(x,y) \sim iH_j^{\pm}(K)\,e^{-Ky\mp iKx} \quad \text{as } x\to\pm\infty \quad (j = 1\sim 4) \tag{6}$$

$$\text{Radiation wave:} \quad \zeta_j^{\pm} = -iKX_jH_j^{\pm}(K)\ \big(\,\equiv X_j\bar{A}_j\,e^{i\varepsilon_j^{\pm}}\,\big) \tag{7}$$

$$\text{Scattered wave:} \quad \zeta_4^{\pm} = iaH_4^{\pm}(K) \tag{8}$$

$$\zeta(x) = a\Big[\,iH_4^+(K) - iK\sum_{j=1}^{3}\frac{X_j}{a}H_j^+(K)\Big]\,e^{-iKx} \quad (\textit{propagating to positive}) \tag{9}$$

$$+a\Big[\,1 + iH_4^-(K) - iK\sum_{j=1}^{3}\frac{X_j}{a}H_j^-(K)\Big]\,e^{+iKx} \quad (\textit{propagating to negative}) \tag{10}$$

Case (1)

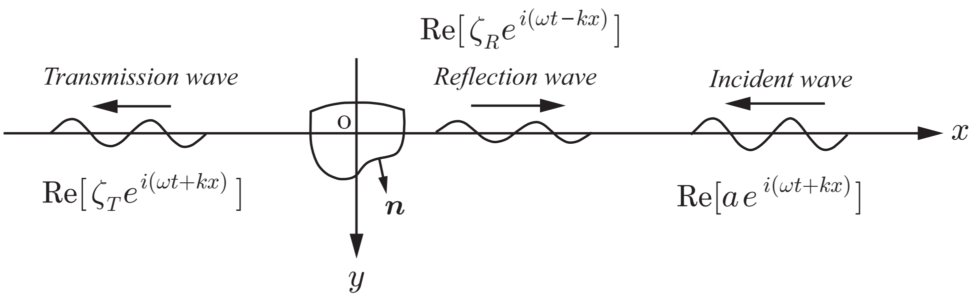

Fig.2 Case of incident wave incoming from the positive x-axis

$$\zeta_R\, e^{-iKx} = a\Big\{\, iH_4^+(K) - iK\sum_{j=1}^{3}\frac{X_j}{a}\, H_j^+(K)\,\Big\}\, e^{-iKx} \tag{11}$$

$$\longrightarrow\quad C_R \equiv \frac{\zeta_R}{a} = \underbrace{iH_4^+(K)}_{\equiv R} - iK\sum_{j=1}^{3}\frac{X_j}{a}\, H_j^+(K) \tag{12}$$

$$R = iH_4^+(K) \tag{13}$$

$$\zeta_T\, e^{+iKx} = a\Big\{\, 1 + iH_4^-(K) - iK\sum_{j=1}^{3}\frac{X_j}{a}\, H_j^-(K)\,\Big\}\, e^{+iKx} \tag{14}$$

$$\longrightarrow\quad C_T \equiv \frac{\zeta_T}{a} = \underbrace{1 + iH_4^-(K)}_{\equiv T} - iK\sum_{j=1}^{3}\frac{X_j}{a}\, H_j^-(K) \tag{15}$$

$$T = 1 + iH_4^-(K) \tag{16}$$

Case (2)

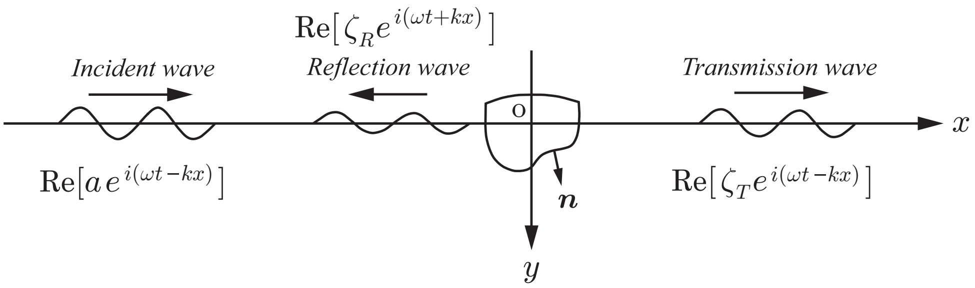

Fig.3 Case of incident wave incoming from the negative x-axis

$$C_R \equiv \frac{\zeta_R}{a} = \underbrace{ih_4^-(K)}_{\equiv R} - iK\sum_{j=1}^{3}\frac{X_j}{a}\, H_j^-(K) \tag{17}$$

$$R = ih_4^-(K) \tag{18}$$

$$C_T \equiv \frac{\zeta_T}{a} = \underbrace{1 + ih_4^+(K)}_{\equiv T} - iK\sum_{j=1}^{3}\frac{X_j}{a}\, H_j^+(K) \tag{19}$$

$$T = 1 + ih_4^+(K) \tag{20}$$

Hydrodynamic Relations Derived with Green's Theorem

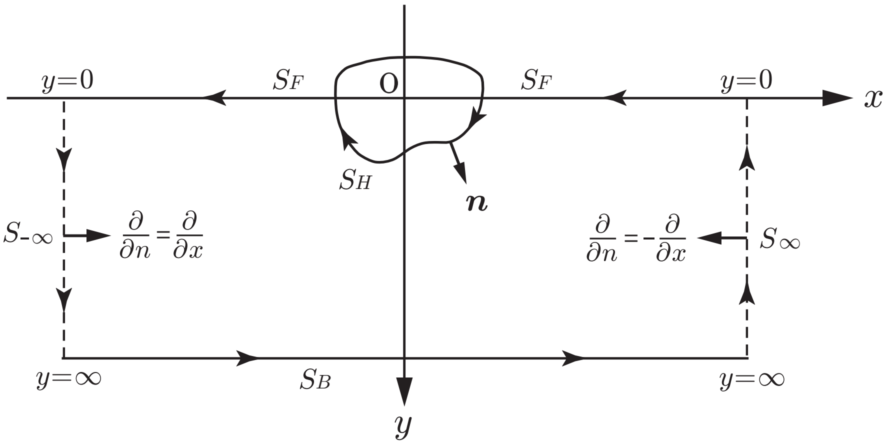

Fig. 3.4 Application of Green's theorem

$$\int_{S_H} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\ell = \frac{1}{2K} \left[\left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right)_{y=0} \right]_{x=-\infty}^{x=+\infty} \tag{1}$$

Table 1 Summary of Some Important Hydrodynamic Relations

	ϕ	ψ	Relations to be obtained
1	φ_i	φ_j	$A_{ij} = A_{ji}\,,\ B_{ij} = B_{ji}$: Symmetry relations in the radiation forces
2	φ_i	$\overline{\varphi}_j$	$B_{ij} = \frac{1}{2}\rho\omega\left\{ H_i^+ \overline{H}_j^+ + H_i^- \overline{H}_j^- \right\}$ Energy conservation in the radiation problem
3	ϕ_D	$\overline{\phi}_D$	$\lvert R \rvert^2 + \lvert T \rvert^2 = 1$ Energy conservation in the diffraction problem
4	ϕ_D	φ_j	$E_j = \rho g a\, H_j^+$ Haskind-Hanaoka-Newman's relation
5	ϕ_D	$\overline{\varphi}_j$	$E_j = \rho g a \left\{ \overline{H}_j^+ R + \overline{H}_j^- T \right\}$
			Relations in 4 and 5 gives $H_j^+ = \overline{H}_j^+ R + \overline{H}_j^- T, \quad (R = iH_4^+,\ T = 1 + iH_4^-)$ For a symmetry body $H_4^\pm = i\, e^{i\varepsilon_H} \cos\varepsilon_H \mp e^{i\varepsilon_S} \sin\varepsilon_S$ Furthermore $\begin{cases} H_3^+ = H_1^+ \ell_w & (\ell_w \text{ is real quantity; the phase is the same}) \\ B_{13} = B_{31} = \ell_w B_{11} & \\ B_{33} = \ell_w^2 B_{11} & (\text{Bessho's relation}) \end{cases}$
6	ϕ_D	ψ_D	$h_4^+ = H_4^-$ Transmission wave (both amplitude & phase) past an asymmetric body is the same irrespective of the direction of incident wave (Bessho's relation)
7	$\overline{\phi}_D$	ψ_D	$h_4^- = \overline{H}_4^+ \dfrac{1 + i\, H_4^-}{1 - i\, \overline{H}_4^-}$ The amplitude of reflection wave by an asymmetric body is also the same irrespective of the direction of incident wave (Bessho's relation)
			Relations of 3 and 7 for a symmetric body gives $\lvert R \pm T \rvert = 1$: Wave energy equally splitting law

In the above, $\phi_D = \varphi_0 + \varphi_4$ and $\overline{\phi}$ denotes the complex conugate of ϕ.