Theory of Ship Waves
(Wave-Body Interaction Theory)

By Professor Masashi KASHIWAGI

Problem Set: No.4
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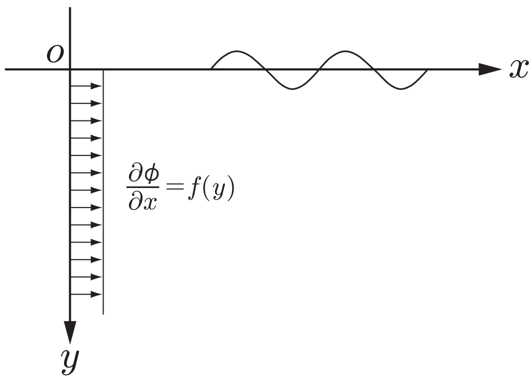
A solution of the velocity potential for a general-shaped body can be obatined numerically by solving an integral equation (which will be explained toward the end of lecture). However, let us consider here an application of Green's theorem to obtain an analytical expression of the velocity potential for a simple (but practical) geometry: that is, a piston-type wavemaker of vertical plate (see Figure below).

For simplicity, the plate is assumed to be zero in thickness and to extend to infinity downwards. With an assumption of sinusoidal motion with circular frequency ω, the velocity potential to be obtained is written as $\Phi = \mathrm{Re}\big[\, \phi(x,y)\, e^{i\omega t} \,\big]$.



Then the bounday condition on the plate is written generally in the form

$$\frac{\partial \phi}{\partial x} = f(y) \quad \text{at } x = 0_{+} \tag{1}$$

In reality, the fluid exists only in the region of $x > 0$. Mathematically, however, we may assume that the fluid is extended to the region of $x < 0$ as well, and the flow can be assumed to be symmetric with respect to $x = 0$. Namely, $\phi(x,y)$ is assumed to be an even function in x due to a symmetric motion in x (like dilatation) of a vertical plate of zero thickness. In this case, the velocity potential and its normal derivative on the plate appearing in Green's theorem may be evaluated as follows:

$$\frac{\partial \phi}{\partial n_Q} = \pm \frac{\partial \phi}{\partial \xi} = f(\eta) \quad \text{at } \xi = 0_{\pm} \tag{2}$$

$$\phi(0_{+}, \eta) = \phi(0_{-}, \eta) \tag{3}$$

Note that the integration valiables are expressed with $Q = (\xi, \eta)$.

With these hints, obtain an explicit analytical expression of the velocity potential by the following procedures:

[1] By applying Green's theorem to the above problem, show mathematical transformations to obtain the following expression of the velocity potential:

$$\phi(x, y) = 2 \int_0^\infty f(\eta)\, G(x, y; 0, \eta)\, d\eta \tag{4}$$

Here $G(x, y; \xi, \eta)$ denotes the free-surface Green function which satisfies the free-surface and radiation conditions, explained in the lecture. Explicitly it can be written as

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \log \frac{r}{r_1} - \frac{1}{\pi} \int_0^\infty \frac{k \cos k(y+\eta) - K \sin k(y+\eta)}{k^2 + K^2}\, e^{-k|x-\xi|}\, dk$$
$$+\, i\, e^{-K(y+\eta) - iK|x-\xi|} \tag{5}$$

More hints: The normal derivative of the Green function on the vertical plate ($\xi = 0_\pm$) may be evaluated as

$$\left.\frac{\partial G(P;Q)}{\partial n_Q}\right|_{\xi=0} = \pm \left.\frac{\partial G(P;Q)}{\partial \xi}\right|_{\xi=0} \tag{6}$$

As can be confirmed from Eq. (5), the values of $\left.\dfrac{\partial G(P;Q)}{\partial \xi}\right|_{\xi=0}$ on both sides of the plate ($\xi = 0_\pm$) are identical.

[2] Obtain an asymptotic expression of the velocity potential at $x \to \infty$ from the result obtained above. (Note that the local wave component will decay rapidly and only the progressive wave component will remain.) Then, identify the Kochin function in the present problem by comparing to the general asymptotic expression of the velocity potential written in the lecture note.

[3] As an example for function $f(\eta)$, let us consider the movement of a piston-type wavemaker. Since $f(\eta)$ is the velocity, we can write

$$f(\eta) = i\omega X, \tag{7}$$

which is a constant (with X motion amplitude and ω circular frequency). In this particular case, perform integrations with respect to η in (4) and obatin the desired result for the velocity potential $\phi(x, y)$.

You may use the following formulae for mathematical transformation:

$$\log \frac{r}{r_1} = -\int_0^\infty \frac{\cos k(y-\eta) - \cos k(y+\eta)}{k}\, e^{-k|x|}\, dk$$
$$= -2 \int_0^\infty \frac{\sin ky \sin k\eta}{k}\, e^{-k|x|}\, dk$$
$$\int_0^\infty e^{ik(y+\eta)}\, d\eta = \left[\frac{1}{ik}\, e^{ik(y+\eta)} \right]_0^\infty = \frac{i}{k}\, e^{iky}$$

The result of [3] must be of the form:

$$\phi(x, y) = i\omega X \left(\frac{2i}{K}\right) e^{-Ky - iKx}$$
$$+\, i\omega X\, \frac{2}{\pi} \int_0^\infty \frac{K(k \cos ky - K \sin ky)}{k^2(k^2 + K^2)}\, e^{-kx}\, dk \qquad \text{for } x > 0$$