Theory of Ship Waves
(Wave-Body Interaction Theory)

Additional Explanation on the Assignment (Problem Set: No. 4)

The analytical integral with respect to η in the following:

$$\phi(x,y) = 2\int_0^\infty f(\eta)\,G(x,y;0,\eta)\,d\eta$$

$$\begin{aligned} G(x,y;0,\eta) = &-\frac{1}{\pi}\int_0^\infty \frac{\sin ky \sin k\eta}{k}\,e^{-k|x|}\,dk \\ &-\frac{1}{\pi}\int_0^\infty \frac{k\cos k(y+\eta) - K\sin k(y+\eta)}{k^2+K^2}\,e^{-k|x|}\,dk \\ &+i\,e^{-K(y+\eta)-iK|x|} \end{aligned}$$

We consider the case of $f(\eta) = i\omega X$ (a piston-type wavemaker), for which we have

$$\begin{aligned} \phi(x,y) &= 2i\omega X\int_0^\infty G(x,y;0,\eta)\,d\eta \\ &= i\omega X\,2i\,e^{-Ky-iK|x|}\int_0^\infty e^{-K\eta}\,d\eta \\ &\quad + i\omega X\Big(-\frac{2}{\pi}\Big)\int_0^\infty \frac{\sin ky}{k}\,e^{-k|x|}\Big\{\int_0^\infty \sin k\eta\,d\eta\Big\}\,dk \\ &\quad + i\omega X\Big(-\frac{2}{\pi}\Big)\int_0^\infty \frac{e^{-k|x|}}{k^2+K^2}\Big\{k\int_0^\infty \cos k(y+\eta)\,d\eta - K\int_0^\infty \sin k(y+\eta)\,d\eta\Big\}\,dk \end{aligned}$$

where we substitute the following formulas:

$$\left\{\begin{aligned} &\int_0^\infty \cos k(y+\eta)\,d\eta = -\frac{1}{k}\sin ky \\ &\int_0^\infty \sin k(y+\eta)\,d\eta = \frac{1}{k}\cos ky \\ &\int_0^\infty \sin k\eta\,d\eta = \frac{1}{k}\,, \quad \int_0^\infty e^{-K\eta}\,d\eta = \frac{1}{K} \end{aligned}\right.$$

Then we can have the following:

$$\begin{aligned} \phi(x,y) &= i\omega X\Big(\frac{2i}{K}\Big)\,e^{-Ky-iK|x|} \\ &\quad + i\omega X\Big(-\frac{2}{\pi}\Big)\Big[\int_0^\infty \frac{\sin ky}{k^2}\,e^{-k|x|}\,dk - \int_0^\infty \frac{e^{-k|x|}}{k(k^2+K^2)}\big\{k\sin ky + K\cos ky\big\}\,dk\Big] \end{aligned}$$

Rewriting the result gives the final form as follows:

$$\begin{aligned} \phi(x,y) &= i\omega X\Big(\frac{2i}{K}\Big)\,e^{-Ky-iK|x|} \\ &\quad + i\omega X\frac{2}{\pi}\int_0^\infty \frac{K(k\cos ky - K\sin ky)}{k^2(k^2+K^2)}\,e^{-k|x|}\,dk \end{aligned}$$

Let us consider the same problem in finite-depth water by means of the eigen-function expansion method. By taking the limit of infinite water depth from the result to be obtained, we will be able to derive the result shown in the previous page. Considering this limit would be helpful in understanding the relation between the finite and infinite water-depth cases.

First, the eigenvalues and corresponding eigenfunctions are explained by solving the following problem:

$$\begin{aligned}
&[L] \quad \nabla^2 \phi(x,y) = 0 \\
&[F] \quad \frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on } y = 0 \\
&[B] \quad \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = h \\
&[R] \quad \text{outgoing waves at } |x| \to \infty
\end{aligned}$$

by means of the variable-separation method.

Let us assume a solution in the form

$$\phi(x,y) = X(x)\,Y(y)$$

Then from Laplace's equation, we have

$$X''Y + XY'' \longrightarrow \frac{x''}{X} = -\frac{Y''}{Y} = \pm k^2$$

i) For the case of $-k^2$

The fundamental solutions of the above are obatined as follows:

$$\begin{aligned}
X'' + k^2 X = 0 \;&\longrightarrow\; X = a_1\, e^{ikx} + a_2\, e^{-ikx} \\
Y'' - k^2 Y = 0 \;&\longrightarrow\; Y = b_1\, e^{-ky} + b_2\, e^{ky}
\end{aligned}$$

Imposing the boundary conditions on $[F]$, $[B]$, and $[R]$ gives the following result:

$$\left.\begin{aligned}
\phi_1(x,y) = A\,\frac{\cosh k_0(y-h)}{\cosh k_0 h}\, e^{-ik_0 x} \quad \text{for } x > 0 \\
\text{where} \qquad k_0 \tanh k_0 h = K
\end{aligned}\right\}$$

ii) For the case of $+k^2$

The fundamental solutions in this case are obatined as follows:

$$\begin{aligned}
X'' - k^2 X = 0 \;&\longrightarrow\; X = c_1\, e^{-kx} + c_2\, e^{kx} \\
Y'' + k^2 Y = 0 \;&\longrightarrow\; Y = d_1 \cos ky + d_2 \sin ky
\end{aligned}$$

A solution satisfying the boundary conditions on $[F]$, $[B]$ and the condition of decaying as $x \to +\infty$ may be given in the form

$$\left.\begin{aligned}
\phi_2(x,y) = B\,\frac{\cos k_n(y-h)}{\cos k_n h}\, e^{-k_n x} \quad \text{for } x > 0 \\
\text{where} \qquad k_n \tan k_n h = -K
\end{aligned}\right\}$$

Therefore, as the sum of the cases of i) and ii) we have an expression for the velocity potential in terms of the eigenfunctions for the finite water depth as follows:

$$\phi(x,y) = A\,\frac{\cosh k_0(y-h)}{\cosh k_0 h}\, e^{-ik_0 x} + \sum_{n=1}^{\infty} B_n\, \frac{\cos k_n(y-h)}{\cos k_n h}\, e^{-k_n x} \tag{1}$$

As the next step, the boundary condition on the wavemaker

$$\frac{\partial \phi}{\partial x} = f(y) = i\omega X \quad \text{on } x = 0 \tag{2}$$

must be satisfied, which may be accomplished by applying the orthogonality in a system of eigenfunctions in the depth-wise direction. Namely, from (1) and (2) we can obtain the relation:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = -Aik_0 \frac{\cosh k_0(y-h)}{\cosh k_0 h} - \sum_{n=1}^{\infty} B_n k_n \frac{\cos k_n(y-h)}{\cos k_n h} = i\omega X \tag{3}$$

Multiplying both sides of (3) by each component in the orthogonal function system, we may have simultaneous equations for the unknowns. In this process, the following results (the orthogonal properties) will be used:

$$\int_0^h \frac{\cosh k_0(y-h)}{\cosh k_0 h}\, dy = \frac{\tanh k_0 h}{k_0} = \frac{K}{k_0^2} \tag{4}$$

$$\begin{aligned}\int_0^h \frac{\cosh^2 k_0(y-h)}{\cosh^2 k_0 h}\, dy &= \frac{1}{\cosh^2 k_0 h}\int_0^h \frac{1+\cosh 2k_0(y-h)}{2}\, dy \\ &= \frac{1}{2k_0 \cosh^2 k_0 h}\big(k_0 h + \sinh k_0 h \cosh k_0 h \big) \\ &= \frac{1}{2k_0^2}\Big\{ h(k_0^2 - K^2) + K \Big\}\end{aligned} \tag{5}$$

Substituting $k_0 \to ik_n$ in above results, we have the followings:

$$\int_0^h \frac{\cos k_n(y-h)}{\cos k_n h}\, dy = \frac{\tan k_n h}{k_n} = -\frac{K}{k_n^2} \tag{6}$$

$$\int_0^h \frac{\cos^2 k_n(y-h)}{\cos^2 k_n h}\, dy = -\frac{1}{2k_n^2}\Big\{ -h(k_n^2 + K^2) + K \Big\} \tag{7}$$

Note that other integrals are zero due to orthogonality of the functions considered here; that is,

$$\int_0^h \frac{\cosh k_0(y-h)}{\cosh k_0 h}\frac{\cos k_n(y-h)}{\cos k_n h}\, dy = 0 \tag{8}$$

With these relations, multiplying both sides of (3) by $\cosh k_0(y-h)/\cosh k_0 h$, integrating from 0 to h with respect to y, and using (4), (5), and (8), we have

$$\begin{aligned} i\omega X K &= -Aik_0 \frac{1}{2}\Big\{ h(k_0^2 - K^2) + K \Big\} \\ \longrightarrow \quad A &= i\omega X \Big(\frac{2i}{k_0}\Big)\frac{K}{K + h(k_0^2 - K^2)}\end{aligned} \tag{9}$$

Likewise, multiplying both sides of (3) by $\cos k_m(y-h)/\cos k_m h$ and integrating from 0 to h with respect to y, we have

$$\begin{aligned} i\omega X(-K) &= B_n k_n \frac{1}{2}\Big\{ -h(k_n^2 + K^2) + K \Big\} \\ \longrightarrow \quad B_n &= -i\omega X \Big(\frac{2}{k_n}\Big)\frac{K}{K - h(k_0^2 + K^2)}\end{aligned} \tag{10}$$

Substituting these results in (1) gives the following result:

$$\phi(x,y) = i\omega X \Big(\frac{2i}{k_0}\Big) C_0 \frac{\cosh k_0(y-h)}{\cosh k_0 h}\, e^{-ik_0 x} - i\omega X \sum_{n=1}^{\infty} \frac{2}{k_n} C_n \frac{\cos k_n(y-h)}{\cos k_n h}\, e^{-k_n x} \tag{11}$$

where
$$C_0 = \frac{K}{K + h(k_0^2 - K^2)}, \quad C_n = \frac{K}{K - h(k_n^2 + K^2)}$$

Let us consider the limit of $h \to \infty$ from the above result. In considering this limit, we note that $\tanh k_0 h \to 1$ and thus

$$k_0 \longrightarrow K\,, \quad C_0 \longrightarrow 1\,, \quad \frac{\cosh k_0(y-h)}{\cosh k_0 h} \longrightarrow e^{-Ky}$$

On the other hand, concerning the second term of (11), discrete wavenumber k_n becomes a continuous variable and

$$\frac{\cos k_n(y-h)}{\cos k_n h} = \cos k_n y + \tan k_n h \sin k_n y \longrightarrow \frac{1}{k}\big(k\cos ky - K\sin ky\big)$$

$$C_n \longrightarrow \frac{K}{-h(k_n^2+K^2)} = \frac{K k_n}{-k_n h(k_n^2+K^2)} \longrightarrow -\frac{1}{\pi}\frac{K\,dk}{k^2+K^2}$$

Here, since $k_n h \sim n\pi$, k_n/n has been understood as $\varDelta k$, which is the same limiting operation as in obtaining the Fourier transform from the complex form of Fourier series. The summation with respect to n must be converted as an integral with respect to k. In this way, we have the following expression from (11) as the limit of infinite water depth:

$$\begin{aligned}\phi(x,y) = {}& i\omega X\Big(\frac{2i}{K}\Big)\,e^{-Ky-iKx}\\ &+ i\omega X\frac{2}{\pi}\int_0^\infty \frac{K(k\cos ky - K\sin ky)}{k^2(k^2+K^2)}\,e^{-kx}\,dk \quad \text{for } x>0\end{aligned}$$

which is the same as the result derived before by using Green's theorem.

Another Solution Method

With the assumption of infinite water depth from the beginning, let us consider the same problem using a special form of eigenfunction expansion method.

By means of the variable-separation method, a general solution of the velocity potential satisfying [F], [B] and [R] may be written in the form

$$\begin{aligned}\phi(x,y) = {}& A\,e^{-Ky-iKx}\\ &+ \int_0^\infty B(k)\big(\,k\cos ky - K\sin ky\,\big)\,e^{-kx}\,dk \quad \text{for } x>0\end{aligned} \tag{12}$$

where A and $B(k)$ are unknowns to be determined.

To satisfy the body boundary condition, differentiation of (12) with respect to x is considered on $x=0$, which can readily be given as follows:

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= f(y) = i\omega X\\ &= -iKA\,e^{-Ky} - \int_0^\infty B(k)\,k\big(\,k\cos ky - K\sin ky\,\big)\,dk\end{aligned} \tag{13}$$

Here, as will be proven later, we note that the Fourier integral theorem provides the following identity for arbitrary function $f(y)$:

$$\begin{aligned}f(y) = {}& 2K\,e^{-Ky}\int_0^\infty f(\eta)\,e^{-K\eta}\,d\eta\\ &+ \frac{2}{\pi}\int_0^\infty dk\int_0^\infty f(\eta)\,\frac{(k\cos ky - K\sin ky)(k\cos k\eta - K\sin k\eta)}{k^2+K^2}\,d\eta\end{aligned} \tag{14}$$

Therefore, comparing (13) with (14), we can find the following relations to be satisfied:

$$-\,iA = 2\int_0^\infty i\omega X\,e^{-K\eta}\,d\eta = i\omega X\frac{2}{K} \tag{15}$$

$$-\,B(k)\,k = \frac{2}{\pi}\int_0^\infty i\omega X\,\frac{(k\cos k\eta - K\sin k\eta)}{k^2+K^2}\,d\eta = \frac{2}{\pi}\big(i\omega X\big)\frac{1}{k^2+K^2}\Big(-\frac{K}{k}\Big) \tag{16}$$

Thus we can determine the unknowns as follows:

$$\left.\begin{aligned} A &= i\omega X\,\frac{2\,i}{K} \\ B(k) &= i\omega X\,\frac{2}{\pi}\,\frac{K}{k^2(k^2+K^2)} \end{aligned}\right\} \tag{17}$$

Substituting these results in (12), we can obtain the solution in the form

$$\begin{aligned} \phi(x,y) = &\, i\omega X\Big(\frac{2i}{K}\Big)\,e^{-Ky-iKx} \\ &+ i\omega X\,\frac{2}{\pi}\int_0^\infty \frac{K(\,k\cos ky - K\sin ky\,)}{k^2(k^2+K^2)}\,e^{-kx}\,dk \end{aligned} \tag{18}$$

which is of course the same as the result obtained by another method.

Proof of Equation (14)

$$\begin{aligned} f(y) = &\, 2K\,e^{-Ky}\int_0^\infty f(\eta)\,e^{-K\eta}\,d\eta \\ &+ \frac{2}{\pi}\int_0^\infty dk\int_0^\infty f(\eta)\frac{(k\cos ky - K\sin ky)(k\cos k\eta - K\sin k\eta)}{k^2+K^2}\,d\eta \end{aligned} \tag{19}$$

may be proven from the Fourier integral theorem (Fourier transform). The proof written below was actually obtained by myself about 30 years ago when I was a graduate student.

According to the Fourier transform, we have the following identity:

$$f(y) = \frac{1}{2\,\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(\eta)\,e^{ik(y-\eta)}\,dk\,d\eta = \frac{1}{\pi}\int_0^\infty dk\int_{-\infty}^\infty f(\eta)\cos k(y-\eta)\,d\eta \tag{20}$$

With understanding that $f(y)$ is defined for $y>0$ and η is interpreted as $|\eta|$ (or $f(y)$ is assumed to be even in y), (20) can be written as the Fourier cosine transform:

$$f(y) = \frac{2}{\pi}\int_0^\infty dk\int_0^\infty f(\eta)\cos k(y-\eta)\,d\eta \tag{21}$$

Here we can transform $\cos k(y-\eta)$ as follows:

$$\begin{aligned} \cos k(y-\eta) &= \frac{1}{k^2+K^2}\Big\{k^2\cos k\eta\cos ky + K^2\sin k\eta\sin ky\Big\} \\ &\quad + \frac{1}{k^2+K^2}\Big\{K^2\cos k\eta\cos ky + k^2\sin k\eta\sin ky\Big\} \\ &= \frac{1}{k^2+K^2}\Big\{k^2\cos k\eta\cos ky + K^2\sin k\eta\sin ky\Big\} \\ &\quad + \frac{K^2}{k^2+K^2}\cos k(y+\eta) + \sin k\eta\sin ky \end{aligned} \tag{22}$$

Furthermore

$$\begin{aligned} &\frac{1}{k^2+K^2}\Big\{(k\cos ky - K\sin ky)(k\cos k\eta - K\sin k\eta)\Big\} \\ &\quad = \frac{1}{k^2+K^2}\Big\{k^2\cos k\eta\cos ky + K^2\sin k\eta\sin ky\Big\} - \frac{kK}{k^2+K^2}\sin k(y+\eta) \end{aligned} \tag{23}$$

Therefore we can obtain the following relation:

$$\begin{aligned} \cos k(y-\eta) = &\,\frac{(k\cos ky - K\sin ky)(k\cos k\eta - K\sin k\eta)}{k^2+K^2} \\ &+ \frac{K}{k^2+K^2}\Big\{k\sin k(y+\eta) + K\cos k(y+\eta)\Big\} + \sin k\eta\sin ky \end{aligned} \tag{24}$$

Then regarding the integrals with respect to k, we will use the following results (which can be proven with the inverse Laplace transform or the Fourier transform):

$$\left.\begin{aligned} \frac{2}{\pi}\int_0^\infty \frac{k}{k^2+K^2}\sin k(y+\eta)\,dk &= \; e^{-K(y+\eta)} \\ \frac{2}{\pi}\int_0^\infty \frac{K}{k^2+K^2}\cos k(y+\eta)\,dk &= \; e^{-K(y+\eta)} \end{aligned}\right\} \tag{25}$$

$$\int_0^\infty \sin k\eta \sin ky\,dk = \frac{1}{2}\int_0^\infty \Big\{ \cos k(y-\eta) - \cos k(y+\eta) \Big\} dk = 0 \tag{26}$$

Summarizing the results shown above, we can finally obtain

$$\begin{aligned} f(y) &= \int_0^\infty f(\eta)\left[\frac{2}{\pi}\int_0^\infty \cos k(y-\eta)\,dk\right] d\eta \\ &= \frac{2}{\pi}\int_0^\infty dk \int_0^\infty f(\eta)\frac{(k\cos ky - K\sin ky)(k\cos k\eta - K\sin k\eta)}{k^2+K^2}\,d\eta \\ &\quad + 2K\,e^{-Ky}\int_0^\infty f(\eta)\,e^{-K\eta}\,d\eta \end{aligned} \tag{27}$$

which is the desired indentity for $f(y)$ shown as (19).

Note on the Velocity Potentials due to Source and Doublet

It has beeen shown before that the general solution of the 2D Laplace equation is written in the polar coordinate system as follows:

$$\Phi = C\log r + \sum_{n=1}^{\infty}\frac{1}{r^n}\Big\{ A_n\cos n\theta + B_n \sin n\theta \Big\} \tag{28}$$

We note that higher-order fundamental solutions can be obtained by successive differentiations of the principal solution, $C\log r$, with respect to x and y. Let us show this fact.

$$\left.\begin{aligned} &x = r\cos\theta,\;\; y = r\sin\theta \\ &r = \sqrt{x^2+y^2},\;\; \theta = \tan^{-1}\frac{y}{x} = \frac{\pi}{2} - \tan^{-1}\frac{x}{y} \end{aligned}\right\} \tag{29}$$

Thus we have the following formulae for differentiation.

$$\left\{\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \end{aligned}\right. \tag{30}$$

The velocity potential of a point source is the principal solution given as

$$\phi_0 = C\log r \tag{31}$$

Then by differentiation we have the following results:

$$\phi_{dx} = -\frac{\partial\phi_0}{\partial x} = -C\,\frac{\cos\theta}{r} \tag{32}$$

$$\phi_{dy} = -\frac{\partial\phi_0}{\partial y} = -C\,\frac{\sin\theta}{r} \tag{33}$$

These velocity potentials are known as those of doublet with the axis in the x and y directions, respectively. In the same way, we can obatin the second-order terms as follows.

$$\frac{\partial^2 \phi_0}{\partial x^2} = -\frac{\partial \phi_{dx}}{\partial x} = C\left\{ \frac{\cos^2\theta}{r^2} - \frac{\sin^2\theta}{r^2} \right\} = C\frac{\cos 2\theta}{r^2} \tag{34}$$

$$\frac{\partial \phi_0}{\partial x \partial y} = -\frac{\partial \phi_{dx}}{\partial y} = C\left\{ \frac{\sin\theta\cos\theta}{r^2} + \frac{\cos\theta\sin\theta}{r^2} \right\} = C\frac{\sin 2\theta}{r^2} \tag{35}$$

In summary, the free-surface Green function, $G(x, y; \xi, \eta)$, is the velocity potential due to a point source with unit strength satisfying the linear free-surface boundary condition, and thus the normal differentialtion of the Green function

$$\frac{\partial}{\partial n_Q} G(x, y; \xi, \eta)$$

appearing in the Green's theorem

$$\phi(\mathrm{P}) = \int_{S_H} \left\{ \frac{\partial \phi(\mathrm{Q})}{\partial n_Q} - \phi(\mathrm{Q}) \frac{\partial}{\partial n_Q} \right\} G(\mathrm{P}; \mathrm{Q})\, ds(\mathrm{Q})$$

can be hydrodynamically understood as the velocity potential of doublet with the axis in the normal direction. In other words, the velocity potential describing the flow generated by a general-shaped body can be obtained by a distribution of the source and the doublet with normal axis along the surface of the body.