Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Section 2.1

Meaning of Eq. (2.2)

Let us consider the total amount of flux Q across the boundary C of the fluid (in the 2D problem) denoted as S.

$$Q = \int_C \frac{\partial \phi}{\partial n}\, d\ell = \int_C \boldsymbol{n} \cdot \nabla \phi \, d\ell = \iint_S \nabla \cdot \nabla \phi \, dS = \iint_S \nabla^2 \phi \, dS$$

If there is no singularity (like source), $Q = 0$ due to the conservation of mass. However, if there is a source within S, the flux Q must be not zero but equal to the amount coming from the point of source; which is assumed to be unit in the present case. Thus when the source is located at $(x, y) = (0, \eta)$,

$$Q = \iint_S \nabla^2 \phi \, dS = 1 \quad \longrightarrow \quad \nabla^2 \phi = \delta(x)\, \delta(y - \eta)$$

Because, by the definition of the delta function, it follows that

$$\int_{-\epsilon}^{+\epsilon} \delta(x)\, dx = 1, \qquad \int_{\eta-\epsilon}^{\eta+\epsilon} \delta(y - \eta)\, dy = 1$$

Derivation of Laplace Equation in the Polar Coordinates

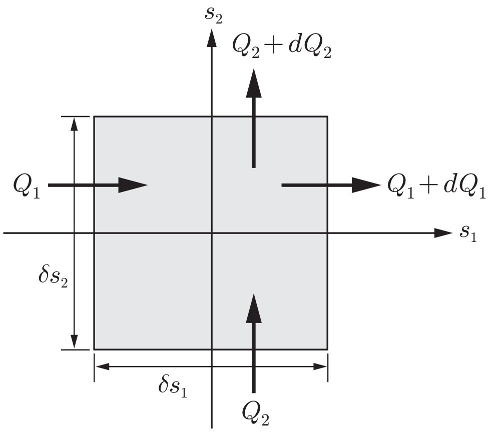

Fig. 1 Flux in $s_1 s_2$ coordinate system

Let us consider the 2D case with the $s_1\, s_2$ coordinate system. By considering the flux in the s_1 direction using the velocity potential, we have the followings:

$$\left.\begin{aligned} Q_1 &= \frac{\partial \phi}{\partial s_1}\, \delta s_2 \\ dQ_1 &= \frac{\partial}{\partial s_1}\left(\frac{\partial \phi}{\partial s_1} \delta s_2\right) \delta s_1 \end{aligned}\right\}$$

Likewise, by considering the flux in the s_2 direction, it follows that

$$\left.\begin{aligned} Q_2 &= \frac{\partial \phi}{\partial s_2}\, \delta s_1 \\ dQ_2 &= \frac{\partial}{\partial s_2}\left(\frac{\partial \phi}{\partial s_2} \delta s_1\right) \delta s_2 \end{aligned}\right\}$$

Since the total net flux must be zero due to the principle of mass conservation, we have the following relation:

$$\frac{\partial}{\partial s_1}\left(\frac{\partial \phi}{\partial s_1} \delta s_2\right) \delta s_1 + \frac{\partial}{\partial s_2}\left(\frac{\partial \phi}{\partial s_2} \delta s_1\right) \delta s_2 = 0 \tag{1}$$

For the case of polar coordinate system, we can find $\delta s_1 = \delta r$ and $\delta s_2 = r \delta \theta$ by calculating $(dx)^2 + (dy)^2$ from the relation:

$$x = r \cos\theta\,, \quad y = r \sin\theta$$

$$\longrightarrow \quad dx = dr \cos\theta - r \sin\theta \, d\theta\,, \quad dy = dr \sin\theta + r \cos\theta \, d\theta$$

$$\longrightarrow \quad (dx)^2 + (dy)^2 = (dr)^2 + (r\, d\theta)^2$$

Then, substituting $\delta s_1 = \delta r$, $\delta s_2 = r\delta\theta$ into Eq. (1) and diving the result by $\delta s_1 \delta s_2$, we have

$$\frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial r} r \delta\theta\right) \delta r + \frac{\partial}{r \partial \theta}\left(\frac{\partial \phi}{r \partial \theta} \delta r\right) r \delta\theta = 0 \quad \longrightarrow \quad \frac{\partial}{r\, \partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Solution Methods for 2-D Laplace Equation

1. Solution in the Cartesian Coordinate System

The x-axis is taken in the horizontal direction and the positive y-axis is taken vertically downward, with the origin on the undisturbed free surface.

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{2}$$

Let us apply the method of separation of variables; that is, the solution is assumed to be given in a form of $\Phi = X(x)Y(y)$. Substituting this into (2), we have

$$X'' Y + X Y'' = 0 \tag{3}$$

Therefore

$$\frac{X''}{X} = -\frac{Y''}{Y} \equiv -k^2 \text{ or } +k^2 \tag{4}$$

$$\longrightarrow \quad \left.\begin{aligned} X'' \pm k^2 X = 0 \\ Y'' \mp k^2 Y = 0 \end{aligned}\right\} \tag{5}$$

Constant k is arbitrary, not necessarily integer. If we require that the solution must be finite at infinity ($y \to \infty$ or $|x| \to \infty$), fundamental solutions of (5) are given as

$$-k^2 \quad \longrightarrow \quad e^{-ky}\Big\{ A\cos kx + B\sin kx \Big\} \tag{6}$$

$$+k^2 \quad \longrightarrow \quad e^{-k|x|}\Big\{ A\cos ky + B\sin ky \Big\} \tag{7}$$

Here A and B are constants, possibly functions of parameter k, and thus a general solution may be written in the form

$$-k^2 \quad \longrightarrow \quad \Phi = \int_0^\infty e^{-ky}\Big\{ A(k)\cos kx + B(k)\sin kx \Big\}\, dk \tag{8}$$

$$+k^2 \quad \longrightarrow \quad \Phi = \int_0^\infty e^{-k|x|}\Big\{ A(k)\cos ky + B(k)\sin ky \Big\}\, dk \tag{9}$$

2. Solution in the Polar Coordinate System

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 \tag{10}$$

Applying the method of separation of variables, the solution can be assumed to be of the form $\Phi = R(r)\Theta(\theta)$. Substituting this into (10), we have the following:

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\frac{1}{R} = -\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} \equiv n^2 \tag{11}$$

Therefore

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 R = 0 \tag{12}$$

$$\frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0 \tag{13}$$

Here we should note that the parameter n introduced in (11) must be integer, because fundamental solutions with respect to θ to be given by (13) must be periodic (the values for $\theta = 0$ and $\theta = 2\pi$ must be the same). This situation is different from the case considered in the Cartesian coordinate system.

The elementary solutions for (12) and (13) are given by

$$R = \{ r^n \, , \ r^{-n} \} \, , \qquad \Theta = \{ \cos n\theta \, , \ \sin n\theta \} \tag{14}$$

If we require regularity of a solution at infinity ($r \to \infty$), a general solution can be written as

$$\Phi = \sum_{n=1}^{\infty} \frac{1}{r^n} \Big\{ A_n \cos n\theta + B_n \sin n\theta \Big\} \tag{15}$$

For the case of $n = 0$, the solution will be independent of θ. (The possibility of $\Theta = \theta$ which may be obtained from (13) should be excluded from the condition of periodicity.) In this case,

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) = 0 \ \longrightarrow \ r \frac{dR}{dr} = C \ \longrightarrow \ R = C \log r \tag{16}$$

Thus we have the following equation as a general solution:

$$\Phi = C \log r + \sum_{n=1}^{\infty} \frac{1}{r^n} \Big\{ A_n \cos n\theta + B_n \sin n\theta \Big\} \tag{17}$$

3. Relations of Solutions

There must be some relations between (8) and (17), because both are solutions for the same 2-D Laplace equation. In order to see those relations, let us consider the following integral:

$$\begin{aligned} I &= \int_0^\infty k^{n-1} \, e^{-ky} \big(\cos kx + i \sin kx \big) \, dk \\ &= \int_0^\infty k^{n-1} \, e^{-ky+ikx} \, dk = \int_0^\infty k^{n-1} \, e^{-k(y-ix)} \, dk \end{aligned} \tag{18}$$

Introducing the relations $y = r\cos\theta$, $x = r\sin\theta$, we can perform the integral with respect to k as follows:

$$\begin{aligned} I &= \int_0^\infty k^{n-1} \, e^{-kr\exp(-i\theta)} \, dk = \frac{e^{i\theta}}{r} (n-1) \int_0^\infty k^{n-2} \, e^{-kr\exp(-i\theta)} \, dk \\ &= \left(\frac{e^{i\theta}}{r} \right)^{n-1} (n-1)! \int_0^\infty e^{-kr\exp(-i\theta)} \, dk = \left(\frac{e^{i\theta}}{r} \right)^n (n-1)! \end{aligned} \tag{19}$$

Therefore, separating the real and imaginary parts, we have

$$\left. \begin{aligned} \int_0^\infty k^{n-1} \, e^{-ky} \cos kx \, dk = (n-1)! \, \frac{\cos n\theta}{r^n} \\ \int_0^\infty k^{n-1} \, e^{-ky} \sin kx \, dk = (n-1)! \, \frac{\sin n\theta}{r^2} \end{aligned} \right\} \tag{20}$$

As the next, we will consider the following integral as the case of $n = 0$

$$J = \int_0^\infty \frac{1}{k} \, e^{-ky} \cos kx \, dk \tag{21}$$

Then, differentiating with respect to y, we have

$$\frac{dJ}{dy} = - \int_0^\infty e^{-ky} \cos kx \, dk = - \frac{y}{x^2 + y^2}$$

$$\begin{aligned} J &= - \int_0^y \frac{\eta}{x^2 + \eta^2} \, d\eta + C \\ &= - \frac{1}{2} \log(x^2 + y^2) + C(x) = - \log r + C \end{aligned}$$

Namely

$$\int_0^\infty \frac{1}{k}\, e^{-ky} \cos kx\, dk = -\log r + C \tag{22}$$

Equations (20) and (22) are desired relations between the solutions in the Cartesian and polar coordinate systems.

4. Application of Fourier Transform

One-dimensional Fourier transform is defined in the form:

$$\left.\begin{aligned} f^*(k,y) &= \int_{-\infty}^{\infty} f(x,y)\, e^{-ikx}\, dx \\ f(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k,y)\, e^{ikx}\, dk \end{aligned}\right\} \tag{23}$$

Similarly two-dimensional Fourier transform can be expressed as

$$\left.\begin{aligned} f^{**}(k,\ell) &= \iint_{-\infty}^{\infty} f(x,y)\, e^{-ikx-i\ell y}\, dxdy \\ f(x,y) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} f^{**}(k,\ell)\, e^{ikx+i\ell y}\, dkd\ell \end{aligned}\right\} \tag{24}$$

Note the following relations in the Fourier transform:

$$\int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = 1 \tag{25}$$

$$\int_{-\infty}^{\infty} \frac{df(x)}{dx}\, e^{-ikx}\, dx = \Big[f(x)\, e^{-ikx} \Big]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = ikf^*(k) \tag{26}$$

Let us consider a fundamental solution, corresponding to a hydrodynamic source with unit strength situated at $(x,y)=(0,\eta)$, which satisfies

$$\nabla^2\phi = \delta(x)\delta(y-\eta) \tag{27}$$

Applying the Fourier transform with respect to x to (27) and noting (25) and (26), we have

$$\frac{d^2\phi^*}{dy^2} - k^2\phi^* = \delta(y-\eta) \tag{28}$$

A general solution of (28), valid except at $y=\eta$, is given as

$$\phi^*(k,y) = A\, e^{|k|y} + B\, e^{-|k|y} \tag{29}$$

Unknown coefficients, A and B, must be determined from boundary conditions and by taking account of the singularity at $y=\eta$. First, to avoid the singularity at $y=\eta$ for a moment, we divide the region into the lower (denoted as Region 1) and upper (Region 2) parts, separated at point $y=\eta$. Since the region is considered unbounded (no outer boundaries), the solution must be zero at both infinities ($y\to\pm\infty$). Thus the solutions in the lower (written as ϕ_1^*) and upper (ϕ_2^*) regions are given as

$$\phi_1^* = A\, e^{|k|y}\,, \quad \phi_2^* = B\, e^{-|k|y} \tag{30}$$

(Here the positive y-axis is taken vertically upward.)

To take account of the singularity at $y=\eta$, let us integrate (27) over a small region crossing $y=\eta$; i.e. $\eta-\epsilon<y<\eta+\epsilon$ (where ϵ is assumed to very small). The result takes a form

$$\left[\frac{d\phi^*}{dy}\right]_{\eta-\epsilon}^{\eta+\epsilon} - k^2\int_{\eta-\epsilon}^{\eta+\epsilon}\phi^*\, dy = \int_{\eta-\epsilon}^{\eta+\epsilon}\delta(y-\eta)\, dy = 1 \tag{31}$$

This relation can be satisfied, provided that

$$\phi_2^* = \phi_1^* \,, \quad \frac{d\phi_2^*}{dy} - \frac{d\phi_1^*}{dy} = 1 \quad \text{at } y = \eta \tag{32}$$

Substituting (30) in (32) gives the followings:

$$\left.\begin{aligned} B\, e^{-|k|\eta} - A\, e^{|k|\eta} = 0 \\ -|k| \big(B\, e^{-|k|\eta} + A\, e^{|k|\eta} \big) = 1 \end{aligned}\right\} \tag{33}$$

From these we can determine A and B as follows:

$$A = -\frac{1}{2|k|}\, e^{-|k|\eta} \,, \quad B = -\frac{1}{2|k|}\, e^{|k|\eta} \tag{34}$$

Substituting these into (30) gives the desired solution in the form

$$\left.\begin{aligned} \phi_1^* &= -\frac{1}{2|k|}\, e^{|k|(y-\eta)} && \text{for } y - \eta < 0 \\ \phi_2^* &= -\frac{1}{2|k|}\, e^{-|k|(y-\eta)} && \text{for } y - \eta > 0 \end{aligned}\right\} \tag{35}$$

These two can be written as a unified form

$$\phi^*(k, y) = -\frac{1}{2|k|}\, e^{-|k||y-\eta|} \tag{36}$$

As the next step for obtaining the solution in the physical domain, the inverse Fourier transform must be considered, which becomes

$$\begin{aligned} \phi(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ -\frac{1}{2|k|}\, e^{-|k||y-\eta|} \right\} e^{ikx}\, dk = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{|k|}\, e^{-|k||y-\eta| + ikx}\, dk \\ &= -\frac{1}{2\pi} \int_0^{\infty} \frac{1}{k}\, e^{-k|y-\eta|} \cos kx\, dk = -\frac{1}{2\pi} \,\mathrm{Re} \int_0^{\infty} \frac{1}{k}\, e^{ik(x + i|y-\eta|)}\, dk \end{aligned} \tag{37}$$

In relation to this integral, let us consider the following integral:

$$F(r) = \int_0^{\infty} \frac{1}{k}\, e^{ikr}\, dk \tag{38}$$

This integral can be evaluated as follows:

$$\begin{aligned} \frac{dF}{dr} &= i \int_0^{\infty} e^{ikr}\, dk = \left[\frac{1}{r}\, e^{ikr} \right]_0^{\infty} = -\frac{1}{r} \\ &\longrightarrow \quad F(r) = -\log r + C \end{aligned} \tag{39}$$

Therefore we can recast the result of (37) in the form

$$\phi(x, y) = -\frac{1}{2\pi} \,\mathrm{Re}\Big[-\log\big(x + i|y-\eta|\big) \Big] = \frac{1}{2\pi} \log r \,, \quad r = \sqrt{x^2 + (y-\eta)^2} \tag{40}$$

The procedure explained above looks complicated, but it will be useful in understanding derivation of the free-surface Green function.

To see another method using the Fourier transform, let us apply the two-dimensional Fourier transform to the following Laplace equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \delta(x)\delta(y) \tag{41}$$

With relations of (25) and (26), it follows that

$$-\big(\,k^2 + \ell^2\,\big)\phi^{**} = 1 \quad\longrightarrow\quad \phi^{**}(k,\ell) = -\,\frac{1}{k^2+\ell^2} \tag{42}$$

Thus the inverse Fourier transform provides the following expression:

$$\phi(x,y) = -\,\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \frac{1}{k^2+\ell^2}\,e^{ikx+i\ell y}\,dk d\ell \tag{43}$$

Let us first perform the integral with respect to ℓ:

$$I(k) = \int_{-\infty}^{\infty} \frac{e^{i\ell y}}{\ell^2+k^2}\,d\ell \tag{44}$$

Since there are singular points at $\ell = \pm i|k|$, a contour integral taken in the upper half complex plane (for the case of $y > 0$) gives the following result:

$$I(k) = 2\pi i\frac{e^{-|k|y}}{2i|k|} = \frac{\pi}{|k|}\,e^{-|k|y} \quad \text{for } y > 0 \tag{45}$$

Substituting this in (43), we have the result:

$$\phi(x,y) = -\,\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{|k|}\,e^{-|k||y|+ikx}\,dk = -\,\frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{k}\,e^{-k|y|}\cos kx\,dk \tag{46}$$

This is the same as (37) and thus the final expression will be the same as (40), as expected.