Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Section 1.5

Plane Progressive Waves

The free-surface elevation:

$$\zeta = \frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{z=0} = A\cos(\omega t - kx) = \mathrm{Re}\big[\,A\,e^{-ikx}\,e^{i\omega t}\,\big] \tag{1}$$

The phase function $\omega t - kx$, which represents a wave propagating in the positive x-axis, because $f(\omega t - kx)$ satisfies

$$\left.\begin{aligned} \frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} &= \big(\,\omega - ck\,\big)f' = 0 \\ c &= \frac{\omega}{k} > 0 \end{aligned}\right\} \tag{2}$$

Here k is the *wavenumber*, ω is the *circular* (or angular) *frequency* , and c is the *phase velocity.*

Dispersion Relation

The general solution to be obtained from Laplace's equation is assumed to be in a form

$$\phi(x,z,t) = \mathrm{Re}\big[\,Z(z)\,e^{-ikx}\,e^{i\omega t}\,\big] \tag{3}$$

Here it should be noted that $Z(z)$ can be complex. Then a general solution for $Z(z)$ is given by

$$Z(z) = C\,e^{kz} + D\,e^{-kz} \tag{4}$$

where C and D are unknown to be determined from boundary conditions.

The free-surface and bottom boundary conditions for $Z(z)$ are written as follows:

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \phi}{\partial z} = 0 \;\longrightarrow\; -\omega^2 Z - g\frac{dZ}{dz} = 0 \quad \text{on } z = 0 \tag{5}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad\qquad \longrightarrow\; \frac{dZ}{dz} = 0 \quad \text{on } z = h \tag{6}$$

Substituting (4) into (5) and (6) gives the following:

$$\left.\begin{aligned} C(\omega^2 + gk) + D(\omega^2 - gk) = 0 \\ C\,e^{kh} - D\,e^{-kh} = 0 \end{aligned}\right\} \tag{7}$$

We note that both unknowns, C and D, cannot be determined uniquely only from these equations (because both equations above are homogeneous ones). However, in order to have non-trivial solutions, the following relation must hold:

$$\begin{vmatrix} \omega^2 + gk & \omega^2 - gk \\ e^{kh} & -\,e^{-kh} \end{vmatrix} = 0 \tag{8}$$

$$\begin{aligned} &\longrightarrow \qquad -\,e^{-kh}(\omega^2 + gk) -\, e^{kh}(\omega^2 - gk) = 0 \\ &\longrightarrow \qquad gk\big(\,e^{kh} -\, e^{-kh}\big) - \omega^2\big(\,e^{kh} +\, e^{-kh}\big) = 0 \\ &\longrightarrow \qquad k\tanh kh = \frac{\omega^2}{g} \end{aligned} \tag{9}$$

This is the *dispersion relation*, implying that the wavenumber (wavelength) and the frequency (period) are mutually dependent parameters.

It should be noted that we can eliminate just one unknown from (7) and the resultant equation can be written in the form

$$Z(z) = \tilde{C}\,\frac{\cosh k(z-h)}{\cosh kh} \qquad \tilde{C} \equiv 2C\,e^{kh}\cosh kh \tag{10}$$

Mathematically speaking, (9) is the eigen-value equation (the equation for eigen values) and (10) is the associated eigen solution or homogeneous solution.

In order to determine the remaining unknown coefficient in (10), we must specify the free-surface elevation given by (1). From (3) and (10), we have

$$\phi(x,z,t) = \mathrm{Re}\Big[\,\tilde{C}\,\frac{\cosh k(z-h)}{\cosh kh}\,e^{-ikx}\,e^{i\omega t}\Big] \tag{11}$$

$$\longrightarrow \quad \zeta = \frac{1}{g}\frac{\partial\phi}{\partial t}\bigg|_{z=0} = \mathrm{Re}\Big[\,\frac{i\omega}{g}\tilde{C}\,e^{-ikx}\,e^{i\omega t}\Big] \tag{12}$$

By comaring the above with (1), we can determine $\tilde{C}$ as follows:

$$\frac{i\omega}{g}\,\tilde{C} = A \quad \longrightarrow \quad \tilde{C} = \frac{gA}{i\omega} \tag{13}$$

Then the solution can be obtained in the form:

$$\phi = \mathrm{Re}\Big[\,\frac{gA}{i\omega}\frac{\cosh k(z-h)}{\cosh kh}\,e^{-ikx}\,e^{i\omega t}\Big] = \frac{gA}{\omega}\frac{\cosh k(z-h)}{\cosh kh}\sin(\omega t - kx) \tag{14}$$

Approximation of $\tanh kh \simeq 1$ is valid for $kh > \pi$ with error less than 0.4 %. This means that if

$$kh = \frac{2\pi h}{\lambda} > \pi \quad \longrightarrow \quad h > \frac{\lambda}{2} \tag{15}$$

is satisfied, the dispersion relation can be practically the same as that for deep water.

For the deep-water case, several relations become rather simple as follows:

$$k = K = \frac{\omega^2}{g} = \frac{2\pi}{\lambda}, \quad T = \frac{2\pi}{\omega} = \sqrt{\frac{2\pi\lambda}{g}} \simeq 0.8\sqrt{\lambda} \quad (\,\lambda \simeq 1.56T^2\,) \tag{16}$$

$$\begin{aligned}\phi &= \frac{gA}{\omega}\,e^{-kz}\sin(\omega t - kx)\\ &= \mathrm{Re}\Big[\,\frac{gA}{i\omega}\,e^{-kz-ikx}\,e^{i\omega t}\Big] \equiv \mathrm{Re}\big[\,\varphi(x,z)\,e^{i\omega t}\big]\end{aligned} \tag{17}$$

where

$$\varphi(x,z) = \frac{gA}{i\omega}\,e^{-kz-ikx} \tag{18}$$

Amplitude Dispersion Relation in Deep Water

According to the textbook, the following expression for the phase velocity is obtained:

$$c = \sqrt{\frac{g}{k}}\Big(1 + \frac{1}{2}k^2A^2\Big) = c^{(1)}\Big(1 + \frac{1}{2}k^2A^2\Big) \tag{19}$$

If we require $\frac{1}{2}(kA)^2 < 0.005$, a linear wave may be guaranteed and this requirement gives the following estimation:

$$kA < \sqrt{0.01} \quad \longrightarrow \quad \frac{2A}{\lambda} = \frac{H}{\lambda} < \frac{\sqrt{0.01}}{\pi} \simeq \frac{1}{30} \tag{20}$$

Here H/λ is referred to as the wave steepness.

Theory of Ship Waves (Wave-Body Interaction Theory)

Supplementary notes on Section 1.5.4 and 1.6

Real Part of Eq. (1.65)

We assume that the difference in amplitude is also small, represented as $A_2 - A_1 = \delta A$. Substituting this into Eq. (1.65), we can transform as follows:

$$
\begin{aligned}
& A_1 \Big[1 + \frac{A_2}{A_1} \, e^{i(\delta\omega\cdot t - \delta k\cdot x)} \Big] \\
& \quad = A_1 \big\{ 1 + \, e^{i(\delta\omega\cdot t - \delta k\cdot x)} \big\} + \delta A \, e^{i(\delta\omega\cdot t - \delta k\cdot x)} \\
& \quad = A_1 2 \, e^{i\frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x)} \frac{1}{2} \Big\{ \, e^{i\frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x)} \, + \, e^{-i\frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x)} \Big\} + \delta A \, e^{i(\delta\omega\cdot t - \delta k\cdot x)} \\
& \quad = 2A_1 \cos \Big\{ \frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x) \Big\} \, e^{i\frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x)} + \delta A \, e^{i(\delta\omega\cdot t - \delta k\cdot x)}
\end{aligned}
\tag{1}
$$

Therefore, taking the real part of Eq. (1.65), we can obtain the following result:

$$
\begin{aligned}
\eta &= \mathrm{Re} \Big\{ \, A_1 \Big[1 + \frac{A_2}{A_1} \, e^{i(\delta\omega\cdot t - \delta k\cdot x)} \Big] e^{i(\omega_1 t - k_1 x)} \, \Big\} \\
&= 2A_1 \cos \Big\{ \frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x) \Big\} \cos \Big\{ \Big(\omega_1 + \frac{1}{2}\delta\omega\Big) t - \Big(k_1 + \frac{1}{2}\delta k\Big) x \Big\} \\
& \qquad\qquad + \delta A \cos \big\{ (\omega_1 + \delta\omega) t - (k_1 + \delta k) x \, \big\}
\end{aligned}
\tag{2}
$$

Retaining only the leading term of the above equation gives the following approximation:

$$
\eta = 2A_1 \cos \Big\{ \frac{1}{2}(\delta\omega\cdot t - \delta k\cdot x) \Big\} \cos \big(\omega_1 t - k_1 x\big) + O(\delta\omega, \delta k, \delta A)
\tag{3}
$$

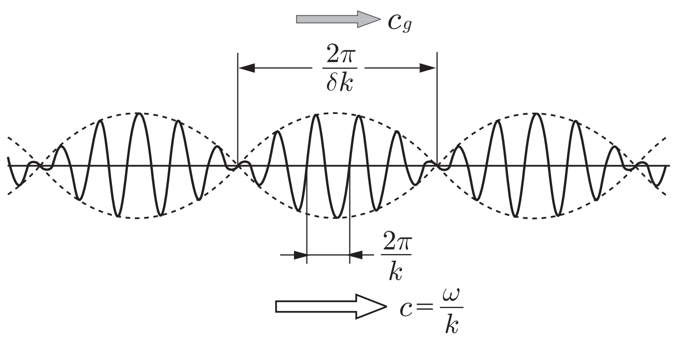

Fig. 1 The amplitude modulation part is an envelope of fundamental carrier waves, and its velocity (group velocity) is given by $\delta\omega/\delta k$.

Calculation of Group Velocity: Eq. (1.69)

The dispersion relation for finite-water depth takes the form

$$
\omega^2 = gk \tanh kh
\tag{4}
$$

The definition of the group velocity is given by

$$
c_g = \frac{d\omega}{dk}
\tag{5}
$$

Taking first the logarithm of both sides of (4) and then differentiating with respect to k, we may have the following

$$
2 \log \omega = \log gk + \log \tanh kh
$$

$$\longrightarrow\ 2\frac{\omega'}{\omega} = \frac{1}{k} + \frac{1}{\tanh kh}\frac{h}{\cosh^2 kh}$$

$$\longrightarrow\ \omega' = \frac{d\omega}{dk} = \frac{1}{2}\frac{\omega}{k}\left\{1 + \frac{kh}{\cosh kh \sinh kh}\right\}$$

$$\longrightarrow\ c_g = \frac{1}{2}c\left\{1 + \frac{2kh}{\sinh 2kh}\right\} \tag{6}$$

<u>Calculation related to Eq. (1.76)</u>

From Eq. (1.58), we can obtain the followings:

$$\phi(x,y) = \frac{ga}{i\omega}\frac{\cosh k(y-h)}{\cosh kh}\, e^{-ikx} \tag{7}$$

$$\frac{\partial\phi}{\partial x} = -\frac{ga}{\omega}k\,\frac{\cosh k(y-h)}{\cosh kh}\, e^{-ikx} \tag{8}$$

$$= -a\omega\,\frac{\cosh k(y-h)}{\sinh kh}\, e^{-ikx} \qquad \longleftarrow \quad \frac{k}{\cosh kh} = \frac{\omega^2}{g\sinh kh} \tag{9}$$

$$\frac{\partial\phi}{\partial y} = \frac{ga}{i\omega}k\,\frac{\sinh k(y-h)}{\cosh kh}\, e^{-ikx} = -ia\omega\,\frac{\sinh k(y-h)}{\sinh kh}\, e^{-ikx} \tag{10}$$

Therefore it follows that

$$\left|\frac{\partial\phi}{\partial x}\right|^2 + \left|\frac{\partial\phi}{\partial y}\right|^2 = (a\omega)^2\,\frac{\cosh^2 k(y-h) + \sinh^2 k(y-h)}{\sinh^2 kh} = (a\omega)^2\,\frac{\cosh 2k(y-h)}{\sinh^2 kh} \tag{11}$$

$$\int_0^h \frac{\cosh 2k(y-h)}{\sinh^2 kh}\, dy = \left[\frac{\sinh 2k(y-h)}{2k\sinh^2 kh}\right]_0^h$$

$$= \frac{\sinh 2kh}{2k\sinh^2 kh} = \frac{\cosh kh}{k\sinh kh} = \frac{g}{\omega^2} \quad \longleftarrow \quad \frac{1}{k\tanh kh} = \frac{g}{\omega^2} \tag{12}$$

Summarizing these results, we can obtain the following result:

$$\frac{1}{4}\rho\int_0^h \left\{\left|\frac{\partial\phi}{\partial x}\right|^2 + \left|\frac{\partial\phi}{\partial y}\right|^2\right\} dy = \frac{1}{4}\rho(a\omega)^2\frac{g}{\omega^2} = \frac{1}{4}\rho g a^2 \tag{13}$$

<u>Calculation related to Eq. (1.78)</u>

From Eq. (7) and Eq. (8), we have

$$i\omega\phi\frac{\partial\phi^*}{\partial x} = -(ga)^2\frac{k}{\omega}\,\frac{\cosh^2 k(y-h)}{\cosh^2 kh} = -(ga)^2\frac{k}{\omega}\,\frac{1}{\cosh^2 kh}\,\frac{1+\cosh 2k(y-h)}{2} \tag{14}$$

$$\int_0^h \frac{1+\cosh 2k(y-h)}{2}\, dy = \frac{1}{2}\left[y + \frac{\sinh 2k(y-h)}{2k}\right]_0^h = \frac{1}{2}\left(h + \frac{\sinh 2kh}{2k}\right)$$

$$= \frac{\sinh 2kh}{4k}\left\{1 + \frac{2kh}{\sinh 2kh}\right\} = \frac{\sinh kh\cosh kh}{2k}\left\{1 + \frac{2kh}{\sinh 2kh}\right\} \tag{15}$$

Therefore, with the dispersion relation $\tanh kh = \omega^2/gk$, we have the following result:

$$\frac{1}{2}\rho\,\mathrm{Re}\int_0^h (i\omega\phi)\frac{\partial\phi^*}{\partial x}\, dy = -\frac{1}{4}\rho(ga)^2\frac{1}{\omega}\frac{\sinh kh}{\cosh kh}\left\{1 + \frac{2kh}{\sinh 2kh}\right\}$$

$$= -\frac{1}{4}\rho(ga)^2\frac{1}{\omega}\frac{\omega^2}{gk}\left\{1 + \frac{2kh}{\sinh 2kh}\right\} = -\frac{1}{4}\rho g a^2\frac{\omega}{k}\left\{1 + \frac{2kh}{\sinh 2kh}\right\} \tag{16}$$