Review of Fundamental Equations

Supplementary notes on Section 1.2 and 1.3

- Introduction of the velocity potential:

  irrotational motion: $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 0$

  identity in the vector analysis: $\nabla \times \nabla\phi \equiv 0$

  $\longrightarrow \;\; \boldsymbol{u} = \nabla\phi$

- Basic conservation principles:

  (1) Conservation of mass

  $\longrightarrow$ Continuity equation $\quad \nabla \cdot \boldsymbol{u} = 0$

  (2) Conservation of momentum

  $\longrightarrow$ Euler's equation (for inviscid fluid) $\quad \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\dfrac{1}{\rho}\nabla p + \boldsymbol{K} \quad (\boldsymbol{K} = g\boldsymbol{k})$

<u>For ideal fluid</u>

From (1) : Laplace's equation $\quad \nabla\cdot\nabla\phi = \nabla^2\phi = 0$

From (2) : Bernoulli's equation $\quad -\dfrac{1}{\rho}(p - p_a) = \dfrac{\partial\phi}{\partial t} + \dfrac{1}{2}\nabla\phi\cdot\nabla\phi - gz$

because of a relation

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u})$$

Euler's equation becomes $\quad \nabla\left[\dfrac{\partial\phi}{\partial t} + \dfrac{1}{2}\nabla\phi\cdot\nabla\phi + \dfrac{p}{\rho} - gz\right] = 0$

<u>Annex</u>

The other identity in the vector analysis: $\quad \nabla\cdot(\nabla\times\boldsymbol{A}) \equiv 0$

From the continuity equation $\nabla\cdot\boldsymbol{u} = 0$

$\longrightarrow \quad \boldsymbol{u} = \nabla\times\boldsymbol{A} \qquad$ ($\boldsymbol{A}$: defined as the vector potential)

For 2-D flows $\boldsymbol{u} = (u, v, 0)$ and thus

the vector potential must be $\boldsymbol{A} = (0, 0, \psi)$

$\longrightarrow \quad u = \dfrac{\partial\psi}{\partial y}, \quad v = -\dfrac{\partial\psi}{\partial x}$

which is known as the stream function for 2-D flows.

<u>Boundary Conditions</u>

(1) Kinematic condition

Fluid particles on a wetted boundary surface, described by $F(x, y, z, t) = 0$, always follow the movement of the boundary surface. Namely, even after a short time interval, the fluid particles remain on the boundary surface. Thus we can write as $F(x + u\Delta t, y + v\Delta t, z + w\Delta t, t + \Delta t) = 0$. Then considering subtraction of these two and applying a Taylor-series expansion with respect to Δt, we may have the following result:

$$\begin{aligned} &F(x + u\Delta t,\, y + v\Delta t,\, z + w\Delta t,\, t + \Delta t) - F(x, y, z, t) \\ &\qquad = u\Delta t\,\frac{\partial F}{\partial x} + v\Delta t\,\frac{\partial F}{\partial y} + w\Delta t\,\frac{\partial F}{\partial z} + \Delta t\,\frac{\partial F}{\partial t} + O\big[(\Delta t)^2\big] = 0 \end{aligned} \tag{1}$$

Then dividing the above by Δt and taking the limit of $\Delta \to 0$, we have the following result:

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} \equiv \frac{DF}{Dt} = 0 \tag{2}$$

With the definition of the velocity potential $\boldsymbol{u} = (u, v, w) = \nabla\phi$, this result can be written as

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla\phi \cdot \nabla F = 0 \qquad \text{on } F = 0 \tag{3}$$

Dividing both sides with $|\nabla F|$ and noting the definition of the normal vector $\boldsymbol{n} = \nabla F/|\nabla F|$, we may have

$$\nabla\phi \cdot \boldsymbol{n} = \frac{\partial \phi}{\partial n} = -\frac{1}{|\nabla F|}\frac{\partial F}{\partial t} \ \left(\equiv V_n \right) \tag{4}$$

(2) Free-surface condition

Provided that the elevation of free surface is expressed as $z = \zeta(x, y, t)$, the kinematic boundary condition is given as follows:

$$F(x, y, z, t) = z - \zeta(x, y, t) = 0 \tag{5}$$

$$\frac{DF}{Dt} = -\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = \zeta(x, y, t) \tag{6}$$

Here we note that both ϕ and ζ are unknown. Thus we need one more boundary condition on the free surface; that is, the dynamic boundary condition which states that the pressure on the free surface is equal to the atmospheric pressure:

$$-\frac{1}{\rho}\left(p - p_a\right) = \frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi - g\zeta = 0 \quad \text{on } z = \zeta(x, y, t) \tag{7}$$

<u>Linearization</u>

Assuming that both ϕ and ζ are of small quantities and retaining only the first-order terms in ϕ and ζ, we may have the followings:

$$-\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial z} + O(\zeta\phi) = 0 \tag{8}$$

$$\zeta = \frac{1}{g}\frac{\partial \phi}{\partial t} + O(\phi^2) = 0 \tag{9}$$

Eliminating ζ from (6) and (7), it follows that

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \phi}{\partial z} + O(\phi^2, \zeta\phi) = 0 \quad \text{on } z = \zeta(x, y, t) \tag{10}$$

Furthermore applying the Taylor-series expansion around the undisturbed free surface ($z = 0$):

$$\phi(x, y, z, t) = \phi(x, y, 0, t) + \zeta\left(\frac{\partial \phi}{\partial z}\right)_{z=0} + \cdots \tag{11}$$

and neglecting higher-order terms resulting from this Taylor expansion, the final result takes the following form:

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = 0 \tag{12}$$