Wave-Body Interaction Theory

(Theory of Ship Waves)

Lecture Notes for Graduate Course

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1. Fundamental Theories on Water Waves

In order to understand theoretically the dynamics of a floating body in waves, it is indispensable to understand the theories related to water waves, that is, the free-surface hydrodynamics. Starting with reviewing the hydrodynamics as a subject of 'mechanics', fundamental knowledge concerning the characteristics of water waves will be summarized in this chapter.

1.1 Continuity Equation and Euler's Equations

Since the hydrodynamics is a subject of 'mechanics' of the fluid, its fundamental equations must be obtained from the conservation laws of physics, such as mass, momentum, and energy, as in the general mechanics. Let us consider first the conservation of mass. We focus our attention on a group of fluid particles (or a material volume of fluid) so that we always examine the same group of particles. Thus we define a volume of fluid $V(t)$ subject to the above restriction, which changes in time. Then by denoting the density of fluid as ρ, the principle of conservation of mass can be written in the form

$$\frac{d}{dt}\iiint_{V(t)} \rho \, dV = 0. \tag{1.1}$$

Next let us consider the principle of conservation of momentum. According to the Newton's second law, the sum of all forces acting on the volume of fluid must be equal to the time rate-of-change of its momentum. Since viscous effects are normally small in the water-wave phenomena, we neglect the viscous shear stress and consider only the normal pressure stress and the gravity force. Then the i-th component ($i = 1$, 2, and 3 correspond to x, y, and z) in the conservation of momentum can be written in the following form

$$\frac{d}{dt}\iiint_{V(t)} \rho \, u_i \, dV = -\iint_{S(t)} p \, n_i \, dS + \iiint_{V(t)} \rho g \, \boldsymbol{k} \, dV. \tag{1.2}$$

Here p denotes the normal pressure acting on the surface S of a prescribed volume of fluid V; n_i the i-th component of the unit normal vector pointing out of V on the surface S; u_i the i-th component of the velocity vector; g the acceleration due to gravity; $\boldsymbol{k}$ is the elementary vector along the z-axis which is taken vertically downwards (in the direction of the gravity force acting).

In the theory of water waves, it is customary to assume that the fluid density ρ is given and unchanged with time. Thus the unknowns in (1.2) are the components of the velocity vector u_i ($i = 1 \sim 3$) and the pressure p; hence the total number of unknowns is four. We can see that (1.1) provides one equation and (1.2) provides three equations. Therefore by solving (1.1) and (1.2) with appropriate boundary and initial conditions applied, the flow of a fluid under consideration can be determined.

In (1.1) and (1.2) especially their left-hand sides, it is supposed that the fluid particles confined by region V are always the same at all times and their movement will be pursued in a Lagrangian way. In order to describe this with a spaced-fixed coordinate system, let us consider a general volume integral of the form

$$I(t) = \iiint_{V(t)} F(\boldsymbol{x}, t) \, dV, \tag{1.3}$$

where F is an arbitrary differentiable scalar function of position vector $\boldsymbol{x}$ and time t. We should bear in mind that the volume of integration is itself a function of time. Therefore the boundary surface S of this volume will change in time and move; let its normal velocity be denoted by U_n.

We consider the variation of (1.3) after a short time interval Δt, which can be written as

$$\Delta I = I(t+\Delta t) - I(t) = \iiint_{V(t+\Delta t)} F(\boldsymbol{x}, t+\Delta t)\, dV - \iiint_{V(t)} F(\boldsymbol{x}, t)\, dV. \tag{1.4}$$

With the Taylor-series expansion, we can write as

$$F(\boldsymbol{x}, t+\Delta t) = F(\boldsymbol{x}, t) + \Delta t \frac{\partial F(\boldsymbol{x}, t)}{\partial t} + O\big[\,(\Delta t)^2\,\big]. \tag{1.5}$$

Neglecting all terms higher than $(\Delta t)^2$, the difference between $V(t+\Delta t)$ and $V(t)$ is a thin volume contained between the adjacent surfaces $S(t+\Delta t)$ and $S(t)$ in the time Δt. Thus, denoting the normal velocity of the boundary surface by U_n, we can write $\Delta V = S(t) U_n \Delta t$. With these taken into consideration, ΔI in (1.4) can be approximated as

$$\begin{aligned}
\Delta I &= \iiint_{V+\Delta V} \left(F + \Delta t \frac{\partial F}{\partial t} \right) dV - \iiint_V F\, dV \\
&= \Delta t \iiint_V \frac{\partial F}{\partial t}\, dV + \iiint_{\Delta V} F\, dV + O\big[\,(\Delta t)^2\,\big] \\
&= \Delta t \iiint_V \frac{\partial F}{\partial t}\, dV + \Delta t \iint_S F\, U_n\, dS + O\big[\,(\Delta t)^2\,\big].
\end{aligned} \tag{1.6}$$

Finally, by dividing both sides by Δt and taking the limit as $\Delta t \to 0$, the time derivative of $I(t)$ defined by (1.3) takes the form

$$\frac{dI}{dt} = \iiint_V \frac{\partial F}{\partial t}\, dV + \iint_S F\, U_n\, dS. \tag{1.7}$$

Equation (1.7) is known as the **transport theorem**. The surface integral in this equation represents the transport of quantity F out of the volume V, as a result of the movement of the boundary.

When V is a material volume, always composed of the same fluid particles, the surface S moves with the same normal velocity as the fluid and hence $U_n = \boldsymbol{u} \cdot \boldsymbol{n} = u_j n_j$. In this case, in terms of the transport theorem (1.7), the conservation of mass (1.1) can be transformed as follows:

$$\begin{aligned}
\frac{d}{dt} \iiint_V \rho\, dV &= \iiint_V \frac{\partial \rho}{\partial t}\, dV + \iint_S \rho\, u_j n_j\, dS \\
&= \iiint_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho\, u_j) \right] dV = 0.
\end{aligned} \tag{1.8}$$

Here Gauss' theorem written in the indicial notation

$$\iiint_V \frac{\partial A_j}{\partial x_j}\, dV = \iint_S A_j n_j\, dS \tag{1.9}$$

has been used in obtaining (1.8).

Since the last integral in (1.8) is evaluated at a fixed instant of time, the distinction that V is a material volume is unnecessary at this stage. Moreover, this volume can be composed of an arbitrary group of fluid particles; hence the integrand itself is equal to zero throughout the fluid. Thus, the volume integration in (1.8) can be replaced by a partial differential equation of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0. \tag{1.10}$$

This is referred to as the **continuity equation**, derived from the conservation of mass.

For an incompressible fluid, the density is constant, and thus (1.10) can be simplified to give the following:

$$\frac{\partial u_j}{\partial x_j} = 0 \quad \text{or} \quad \nabla \cdot \boldsymbol{u} = 0 \tag{1.11}$$

Likewise, by applying the transport theorem (1.7) to the conservation of momentum (1.2), it follows that

$$\iiint_V \left[\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial x_j}(\rho\, u_i\, u_j) \right] dV = \iiint_V \left[-\frac{\partial p}{\partial x_i} + \rho g\, \delta_{i3} \right] dV. \tag{1.12}$$

The right-hand side of this equation has been obtained by using Gauss' theorem (1.9), and δ_{i3} denotes the Kroenecker's delta, equal to 1 if $i = 3$ and 0 otherwise.

Once again the volume of fluid in question is arbitrary; hence (1.12) must hold for the integrands alone, in the form

$$\frac{\partial}{\partial t}\big(\rho u_i\big) + \frac{\partial}{\partial x_j}\big(\rho u_i u_j\big) = -\frac{\partial p}{\partial x_i} + \rho g\, \delta_{i3}. \tag{1.13}$$

Finally, if the derivatives of products on the left-hand side of this equation are expanded with the chain rule, and the continuity equation (1.10) is invoked, we obtain **Euler's equation** in the form

$$\frac{Du_i}{Dt} \equiv \left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) u_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g\, \delta_{i3}. \tag{1.14}$$

We note that this equation is obtained from the conservation of momentum, corresponding to the Newton's second law on the motion equation in the general mechanics. Therefore, the left-hand side of (1.14), denoted as Du_i/Dt, is the time rate-of-change in a coordinate system with the fluid particle and can be interpreted as the acceleration of a material particle of fluid. Thus D/Dt defined in (1.14) is referred to as the **substantial derivative**.

1.2 Potential Flows

In most problems related to water waves, we may assume that the motion of fluid is irrotational; that is, $\nabla \times \boldsymbol{u} = 0$. In other words, the vorticity ($\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$) is zero throughout the fluid. For this particular case, let us consider how (1.11) and (1.14) can be transformed.

According to the formulae in the vector analysis, for an arbitrary scalar function $\Phi(\boldsymbol{x}, t)$, an identity of $\nabla \times \nabla\Phi = 0$ ($\mathrm{rot}\,\mathrm{grad}\,\Phi = 0$) holds. Therefore, if $\nabla \times \boldsymbol{u} = 0$ is satisfied, the velocity vector $\boldsymbol{u}$ can be represented as $\boldsymbol{u} = \nabla\Phi$ in terms of a scalar function Φ, which is called the **velocity potential**, and the flows that can be described with the velocity potential are referred to as the potential flows.

Introducing the velocity potential may at first seem an unnecessary complication, but it is advantageous in a mathematical treatment. The velocity is a vector quantity with three unknown scalar components, whereas the velocity potential is a single scalar unknown from which all three velocity components may be computed.

If $\boldsymbol{u} = \nabla\Phi$ is substituted for the velocity vector in the continuity equation (1.11), it follows that

$$\nabla \cdot \nabla\Phi = \nabla^2\Phi = 0. \tag{1.15}$$

This is the Laplace equation which expresses the conservation of mass for potential flows and provides a partial differential equation as the governing equation to be solved for the function Φ.

We will consider next how Euler's equation (1.14) can be recast for potential flows. For that purpose, we note that the nonlinear term on the left-hand side of Euler's equation can be transformed in the following form with indicial notations

$$\begin{aligned} u_j \frac{\partial}{\partial x_j} u_i &\equiv u_j \partial_j u_i = u_j(\partial_j u_i - \partial_i u_j) + u_j \partial_i u_j \\ &= u_j \varepsilon_{kji} (\nabla \times \boldsymbol{u})_k + \frac{1}{2}\partial_i(u_j u_j) = \frac{1}{2}\nabla(\nabla\Phi \cdot \nabla\Phi), \end{aligned} \tag{1.16}$$

because $\nabla \times \boldsymbol{u} = 0$ is satisfied for potential flows.

Using this relation and substituting $\boldsymbol{u} = \nabla \Phi$ in (1.14), it follows that

$$\nabla \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi + \frac{p}{\rho} - gz \right] = 0. \tag{1.17}$$

Integrating this with respect to the three space variables gives the following expression:

$$p + \rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi - \rho g z = p_0, \tag{1.18}$$

where p_0 is a constant, independent of the space coordinates, which may be taken as the atmospheric pressure, p_a, on the undisturbed still water surface.

Once the velocity potential is determined, the pressure in the fluid can be computed from (1.18); hence (1.18) is referred to as **Bernoulli's pressure equation**. We should note again that Bernoulli's pressure equation is obtained from the conservation of momentum.

1.3 Boundary Conditions

In order to solve the Laplace equation, appropriate boundary conditions must be imposed on the boundaries of the fluid domain. Normally we consider a kinematic boundary condition corresponding to a statement regarding the velocity of the fluid on the boundary. This kinematic boundary condition can always be applied on any boundary surface with a specified geometry and position.

Suppose that the boundary surface is represented with a function $F(\boldsymbol{x}, t) = 0$ in a space-fixed co-ordinate system. Then the kinematic boundary condition can be derived readily by considering the substantial derivative of this function, because the fluid particles on a wetted boundary surface must follow the movement of the boundary surface. Therefore we may write as follows:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla \Phi \cdot \nabla F = 0 \qquad \text{on } F = 0. \tag{1.19}$$

Dividing both sides with $|\nabla F|$ and noting that the normal vector can be computed by $\boldsymbol{n} = \nabla F / |\nabla F|$, we may have

$$\nabla \Phi \cdot \boldsymbol{n} = \frac{\partial \Phi}{\partial n} = -\frac{1}{|\nabla F|} \frac{\partial F}{\partial t} \big(\equiv V_n \big) \qquad \text{on } F = 0. \tag{1.20}$$

This equation provides the boundary condition of Neumann type for the velocity potential, and physically implies that the normal velocity of a fluid particle, denoted by $\partial \Phi / \partial n$, is equal to the normal velocity of the body surface, denoted by V_n. This boundary condition is called the **kinematic condition**.

In a problem of water waves, the water surface (which is referred to as the free surface) is a boundary surface. Suppose that the wave elevation is given by $z = \zeta(x, y, t)$, then the function representing the boundary surface can be expressed as

$$F = z - \zeta(x, y, t) = 0. \tag{1.21}$$

Substituting this in (1.19) gives the kinematic boundary condition in the form

$$\frac{DF}{Dt} = -\frac{\partial \zeta}{\partial t} - \frac{\partial \Phi}{\partial x} \frac{\partial \zeta}{\partial x} - \frac{\partial \Phi}{\partial y} \frac{\partial \zeta}{\partial y} + \frac{\partial \Phi}{\partial z} = 0 \qquad \text{on} \quad z = \zeta(x, y, t). \tag{1.22}$$

Here we note that the wave elevation ζ is also unknown. Thus we need one more boundary condition on the free surface, relating Φ with ζ. In order to realize this requirement, we consider the **dynamic condition** which states that the pressure on the free surface is equal to the atmospheric pressure. Considering $p = p_a$ on $z = \zeta$ in (1.18), we can obtain the desired boundary condition in the form

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi - g\zeta = 0 \qquad \text{on} \quad z = \zeta(x, y, t). \tag{1.23}$$

In principle, by eliminating ζ from (1.22) and (1.23), the boundary condition only for the velocity potential Φ on the free surface may be derived. However, the resulting equation will be a complicated nonlinear one and must be imposed on the exact free surface $z = \zeta$ which is unknown at this stage.

In order to understand fundamental and important characteristics of these equations analytically, we adopt a technique of linearization. That is, we assume that the wave elevation ζ and the velocity potential Φ representing the associated fluid motion are both sufficiently small. Then the derivatives of these quantities will be also of small first order. With these assumptions, we neglect all higher-order terms than $O(\Phi^2)$ in (1.22) and (1.23).

First, from (1.23), the linearized equation for the free-surface elevation is obtained as

$$\zeta = \frac{1}{g}\frac{\partial \Phi}{\partial t} + O(\Phi^2). \tag{1.24}$$

Next, (1.22) may be approximated as

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} + O(\zeta\Phi). \tag{1.25}$$

This equation simply states that the vertical velocities of the free surface and fluid particles are equal, ignoring the small departure of that surface from the horizontal orientation.

Eliminating ζ from (1.24) and (1.25) gives the following equation:

$$\frac{\partial^2 \Phi}{\partial t^2} - g\frac{\partial \Phi}{\partial z} + O(\Phi^2, \zeta\Phi) = 0. \tag{1.26}$$

Although not explicitly shown, this equation must be imposed on the exact free surface $z = \zeta$. To simplify this complication further, we apply to the velocity potential the Taylor-series expansion about the undisturbed free surface $z = 0$, in the form

$$\Phi(x, y, z, t) = \Phi(x, y, 0, t) + \zeta\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} + \cdots. \tag{1.27}$$

Since ζ is assumed small, we can see that the errors induced by applying the boundary condition on $z = 0$ may be higher than $O(\Phi^2)$. Therefore, it is consistent with the linearizations already carried out to impose (1.26) on $z = 0$, and thus the linearized free-surface boundary condition for the velocity potential can be expressed in the following form:

$$\frac{\partial^2 \Phi}{\partial t^2} - g\frac{\partial \Phi}{\partial z} = 0 \qquad \text{on } z = 0. \tag{1.28}$$

The derivation shown above may seem rather complicated, but there exists a more expedient approach, which is suitable for considering the nonlinear free-surface boundary condition for the velocity potential. That approach is to replace the kinematic condition (1.22) by the statement that the substantial derivative of the pressure is zero on the free surface. This is a rather pragmatic mixture of the dynamic and kinematic boundary conditions, because the statement that $Dp/Dt = 0$ on the free surface implies that this is precisely the appropriate moving surface on which the pressure is constant.

Substituting (1.18) for the pressure, we obtain the desired boundary condition in the form

$$\left(\frac{\partial}{\partial t} + \nabla\Phi\cdot\nabla\right)\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi - gz\right) = 0 \qquad \text{on } z = \zeta. \tag{1.29}$$

Working out the indicated derivative gives

$$\frac{\partial^2 \Phi}{\partial t^2} - g\frac{\partial \Phi}{\partial z} + 2\nabla\Phi\cdot\nabla\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\big(\nabla\Phi\cdot\nabla\Phi\big) = 0 \qquad \text{on } z = \zeta. \tag{1.30}$$

It should be noted however that the wave elevation ζ must be evaluated from (1.23). If the technique of linearization described above is applied to (1.30), we can readily obtain the linearized free-surface boundary condition (1.28). Other boundary conditions to be imposed will be explained subsequently when they are needed.

1.4 Principle of Energy Conservation

As a preparation for investigating the characteristics of progressive waves and hydrodynamic forces on a body, the energy and its rate of change with respect to time of ideal fluids will be explained in this section.

According to the knowledge in the general mechanics, the total energy in the fluid is the sum of kinetic and potential energies. Thus, in a prescribed volume V, the total energy is given by the integral

$$E = \iiint_{V(t)} \rho \left[\frac{1}{2} q^2 - gz \right] dV = \iiint_{V(t)} \rho \left[\frac{1}{2} \nabla\Phi \cdot \nabla\Phi - gz \right] dV, \tag{1.31}$$

where $q = |\boldsymbol{u}|$ and the positive z-axis is taken vertically downward.

Next, let us consider the rate of change with respect to time of the total energy (1.31). Allowing the boundary surface S of the volume V to move with the normal velocity U_n and using the transport theorem (1.7), we have

$$\begin{aligned} \frac{dE}{dt} &= \rho \frac{d}{dt} \iiint_{V(t)} \left[\frac{1}{2} q^2 - gz \right] dV \\ &= \rho \iiint_V \frac{\partial}{\partial t} \left[\frac{1}{2} q^2 - gz \right] dV + \rho \iint_S \left[\frac{1}{2} q^2 - gz \right] U_n \, dS. \end{aligned} \tag{1.32}$$

Since z is independent of time, the only contribution to the volume integral in (1.32) is from the kinetic-energy term, which takes the form

$$\frac{\partial}{\partial t} \left[\frac{1}{2} q^2 \right] = \frac{\partial}{\partial t} \left[\frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right] = \nabla\Phi \cdot \nabla \frac{\partial \Phi}{\partial t} = \nabla \cdot \left(\frac{\partial \Phi}{\partial t} \nabla\Phi \right). \tag{1.33}$$

Here the Laplace equation has been used in the last transformation. For the integrand of the surface integral in (1.32), Bernoulli's equation (1.18) can be used to write as

$$\frac{1}{2} q^2 - gz = - \left(\frac{p - p_a}{\rho} + \frac{\partial \Phi}{\partial t} \right). \tag{1.34}$$

Then, substituting these results and applying Gauss' theorem, we can rewrite the volume integral with the surface integral in the form

$$\frac{dE}{dt} = \rho \iint_S \left[\frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial n} - \left(\frac{p - p_a}{\rho} + \frac{\partial \Phi}{\partial t} \right) U_n \right] dS. \tag{1.35}$$

As the boundary surface of a volume of fluid, let us consider the hull surface of a body S_H, the free surface S_F, and a control surface S_C which is at rest and located far from a body. Then the boundary conditions on these surfaces can be written as

$$\left. \begin{aligned} &\text{on } S_C \quad U_n = 0 \\ &\text{on } S_H \quad \frac{\partial \Phi}{\partial n} = U_n = V_n \\ &\text{on } S_F \quad \frac{\partial \Phi}{\partial n} = U_n, \quad p = p_a \end{aligned} \right\} \tag{1.36}$$

Therefore (1.35) can be recast in the form

$$\frac{dE}{dt} = -\iint_{S_H} \left(p - p_a\right) V_n \, dS + \rho \iint_{S_C} \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial n} \, dS. \tag{1.37}$$

If the time average over one cycle is considered, the rate of change of the total energy in the entire fluid (left-hand side) must be zero in time average. The first term on the right-hand side is the work done by a body onto the fluid, because V_n is the normal velocity of the body and the normal vector is defined positive when directing out of the fluid into the body. Denoting this work as W_D, the relation to be obtained from (1.37) for the time average can be expressed as

$$W_D \equiv \overline{-\iint_{S_H} \left(p - p_a\right) V_n \, dS} = \overline{-\rho \iint_{S_C} \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial n} \, dS}. \tag{1.38}$$

This equation can be used in deriving a relation of the damping force on a body with the energy of progressive waves generated by an oscillation of that body. That relation is known as the principle of energy conservation. Of course, if the body is fixed in space ($V_n = 0$) or if no body exists, the left-hand side of (1.38) is zero; thus in these cases, the principle of energy conservation can be written in the form

$$\overline{\iint_{S_C} \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial n} \, dS} = 0. \tag{1.39}$$

1.5 Plane Progressive Waves of Small Amplitude

1.5.1 Phase function and phase velocity

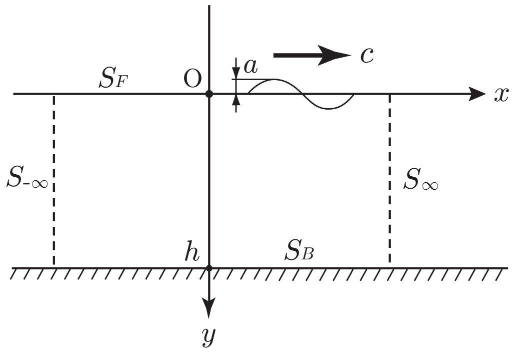


Fig. 1.1 Coordinate system for a plane progressive wave

The simplest solution of the free-surface condition (1.28), which nevertheless has great practical significance, is the plane progressive wave system. As shown in Fig. 1.1, a plane wave is supposed to propagate to the positive x-axis in water of finite constant depth ($y = h$). The motion of fluid is two dimensional in the x-y plane, but the wave propagates only in the x-axis; thus this wave is sometimes called one-dimensional wave.

Let us consider a wave which is sinusoidal in time with circular (or angular or radian) frequency ω; thus the period is given by $T = 2\pi/\omega$. Denoting the amplitude of the wave as a, the elevation of this sinusoidal plane progressive wave can be written as

$$y = \eta(x,t) = a \cos(\omega t - k_0 x), \tag{1.40}$$

where k_0 denotes the wavenumber, the number of waves per unit distance along the x-axis. Thus, in terms of the wavelength λ, it is given by $k_0 = 2\pi/\lambda$.

We should note that the phase function in (1.40) is written as $\omega t - k_0 x$, which represents a wave propagating in the positive x-axis without changing the profile. In order to confirm this, let us introduce a coordinate system (x'-y) moving in the positive x-axis with the same velocity as that of the wave (which is denoted as c). To an observer moving with this velocity, the wave must appear steady-state. Thus, from the following relation

$$\omega t - k_0 x = \omega t - k_0 (x' + ct) = (\omega - k_0 c)t - k_0 x' = -k_0 x',$$

we can see that

$$\omega - k_0 c = 0 \quad \longrightarrow \quad c = \frac{\omega}{k_0} > 0. \tag{1.41}$$

This velocity is the propagation velocity of the wave profile and referred to as the **phase velocity**. With the same discussion, the phase function for a wave propagating in the negative x-axis takes the form of $\omega t + k_0 x$.

These characteristics can be written in another way. We can confirm that an arbitrary function $f(\vartheta)$ with the phase function $\vartheta = \omega t - k_0 x$ satisfies

$$\frac{\partial f}{\partial t} + c\,\frac{\partial f}{\partial x} = \big(\omega - c\,k_0\big)\,f' = 0. \tag{1.42}$$

Namely, a function f satisfying the advection equation of the above form represents the propagation with velocity c in the positive x-axis without changing its profile.

1.5.2 Velocity potential

Let us derive the velocity potential Φ for one-dimensional progressive waves. The governing equation of Φ is the Laplace equation, subject to the boundary conditions on the free surface S_F, the water bottom S_B, and an artificial vertical surface $S_{\pm\infty}$ at $x = \pm\infty$. Those are written as

$$\text{Continuity equation } [L] \qquad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad \text{for } y \geq 0 \tag{1.43}$$

$$\text{Free-surface condition } [F] \qquad \frac{\partial^2 \Phi}{\partial t^2} - g\,\frac{\partial \Phi}{\partial y} = 0 \qquad \text{on } y = 0 \tag{1.44}$$

$$\text{Bottom condition } [B] \qquad \frac{\partial \Phi}{\partial y} = 0 \qquad \text{on } y = h \tag{1.45}$$

The boundary condition on $S_{\pm\infty}$ (which is usually called the radiation condition) is not specified above, but it must be a physically relevant condition. In the present problem, it should state that the wave propagates in the positive x-direction; which can be realized if the phase function takes the form of $f(\omega t - k_0 x)$.

With this consideration, the velocity potential to be obtained is assumed to have the following form:

$$\Phi(x, y, t) = Y(y)\sin(\omega t - k_0 x). \tag{1.46}$$

Substituting this assumed form into the Laplace equation (1.43), we obtain the ordinary differential equation for $Y(y)$ in the form

$$\frac{d^2 Y}{dy^2} - k_0^2 Y = 0. \tag{1.47}$$

The general solution of (1.47) is given as

$$Y(y) = C_1\, e^{k_0 y} + C_2\, e^{-k_0 y}. \tag{1.48}$$

Here C_1 and C_2 are constants to be determined from the boundary conditions on the free surface and the bottom.

Substituting (1.48) in (1.44) and (1.45) gives the following equations:

$$\left.\begin{aligned} C_1(\,\omega^2 + g k_0\,) + C_2(\,\omega^2 - g k_0\,) = 0 \\ C_1\, e^{k_0 h} - C_2\, e^{-k_0 h} = 0 \end{aligned}\right\} \tag{1.49}$$

The necessary and sufficient condition for existence of non-trivial solutions except for $C_1 = C_2 = 0$ is

$$\begin{vmatrix} \omega^2 + g k_0 & \omega^2 - g k_0 \\ e^{k_0 h} & -e^{-k_0 h} \end{vmatrix} = 0. \tag{1.50}$$

Namely

$$\frac{\omega^2}{g} \equiv K = k_0 \tanh k_0 h \tag{1.51}$$

is the required condition; which may be regarded as an equation for the eigen-value to be satisfied between the wavenumber k_0 and the frequency ω.

The corresponding eigen-solution can be obtained as follows. By eliminating one unknown from (1.49) and introducing

$$C_1 \, e^{k_0 h} = C_2 \, e^{-k_0 h} \equiv \frac{1}{2} \, D, \tag{1.52}$$

the eigen-solution can be written in the form

$$\Phi(x, y, t) = D \cosh k_0 (y - h) \sin(\omega t - k_0 x) \tag{1.53}$$

in terms of an unknown coefficient D.

It should be noted that D cannot be determined from (1.49), because the boundary conditions on $[F]$ and $[B]$ are both homogeneous. In order to determine this unknown, the wave elevation on $y = 0$ will be required to be equal to (1.40). Since the linearized free-surface elevation can be computed from (1.24), we have the following relation:

$$\begin{aligned} \eta = \frac{1}{g} \left(\frac{\partial \Phi}{\partial t} \right)_{y=0} &= D \, \frac{\omega}{g} \, \cosh k_0 h \, \cos(\omega t - k_0 x) \\ &= a \, \cos(\omega t - k_0 x). \end{aligned} \tag{1.54}$$

From this, it follows that

$$D = \frac{g a}{\omega \cosh k_0 h}. \tag{1.55}$$

Finally, substituting this result in (1.53) gives the velocity potential for the plane progressive wave in the form

$$\Phi(x, y, t) = \frac{g a}{\omega} \frac{\cosh k_0 (y - h)}{\cosh k_0 h} \sin(\omega t - k_0 x). \tag{1.56}$$

In terms of a complex notation, the above result can be written in the following form

$$\Phi(x, y, t) = \mathrm{Re}\Big[\, \phi(x, y) \, e^{i\omega t} \, \Big], \tag{1.57}$$

$$\phi(x, y) = \frac{g a}{i\omega} \, \frac{\cosh k_0 (y - h)}{\cosh k_0 h} \; e^{-i k_0 x}. \tag{1.58}$$

Here Re in (1.57) means only the real part to be taken. This way of writing, separating the time-dependent term $e^{i\omega t}$ from the spatial part $\phi(x, y)$, is commonly adopted; which makes various calculations easier. However, with this way of writing, the phase function $\omega t - k_0 x$ cannot be indicated in an explicit form. Nevertheless, under the assumption that the time-dependent part is expressed as $e^{i\omega t}$, we must be able to understand that the complex term $e^{-i k_0 x}$ in the spatial part represents a wave propagating in the positive x-axis.

Likewise, we should understand that $e^{+i k_0 x}$ represents a wave propagating in the negative x-axis.

1.5.3 Dispersion relation

We will examine more the meaning of (1.51). Since k_0 and ω are mutually related through (1.51), the phase velocity c defined by (1.41) can be written only with the wavenumber k_0 (or equivalently the wavelength λ), in the form

$$c = \frac{\omega}{k_0} = \sqrt{\frac{g}{k_0} \tanh k_0 h} = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}. \tag{1.59}$$

We can see from (1.59) that the phase velocity varies with the wavelength, and the longer the waves are, the faster they propagate.

In general, ocean waves may be described with a superposition of various components of sinusoidal waves with different wavelength, as can be envisaged from the Fourier-series expansion for an arbitrary function. Each sinusoidal component wave in the ocean propagates with different phase velocity; thus the pattern of ocean waves varies from moment to moment. This characteristic is called the dispersion of waves, and the relation of (1.59) or (1.51), associating the phase velocity with the wavelength, is referred to as the **dispersion relation**.

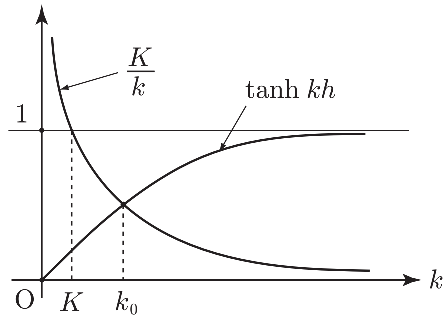


Fig. 1.2 The wavenumber k_0 in finite depth is larger than K in deep water.

The wavenumber k_0 satisfying (1.51) cannot be written in an explicit form as a function of the frequency, because of the hyperbolic tangent. However, since $y = \tanh kh$ is a function of monotonically increasing as schematically shown in Fig. 1.2, we can see that the solution of (1.51), denoted as $k = k_0$, can be obtained uniquely. For the infinite-depth limit ($kh \to \infty$), $\tanh kh = 1$ and thus the wavenumber of progressive wave becomes $k_0 = K$. That is, $k_0 > K$ is always satisfied for waves in finite depth; hence the shallower the water depth, the shorter the wavelength, as compared to the infinite-depth value K.

Considering the two limiting cases for sufficiently deep ($kh = 2\pi h/\lambda \to \infty$) and shallow ($kh = 2\pi h/\lambda \to 0$) in the water depth, it follows from (1.59) that

$$c = \sqrt{\frac{g}{K}} = \sqrt{\frac{g\lambda}{2\pi}} \qquad (\, kh \to \infty\,), \tag{1.60}$$

$$c = \sqrt{gh} \qquad (\, kh \to 0\,). \tag{1.61}$$

In the shallow-water limit (1.61), where the corresponding wave is called the shallow-water wave or the long-wave approximation, the phase velocity depends only on the depth, and the resulting wave motion is no longer dispersive. On the other hand, in order for the deep-water wave to be practically valid, $\tanh kh \sim 1$ must be satisfied with sufficient accuracy. When $kh = 2\pi h/\lambda > \pi$, i.e. $h > \lambda/2$, $\tanh kh \simeq 0.996$ and thus the error is less than 0.4%. We can see from this estimation that large errors may not be caused even if the free-surface wave in finite depth is approximated as the deep-water wave, provided that the water depth is larger than half of the wavelength. This condition looks applicable to substantially almost all waves in the field of ocean engineering. The case of deep water makes various equations explained in this section simpler. Thus some important equations for the deep-water case are summarized below:

$$k\left(=\frac{2\pi}{\lambda}\right) = K = \frac{\omega^2}{g}, \quad T = \frac{2\pi}{\omega} = \sqrt{\frac{2\pi\lambda}{g}} \simeq 0.8\sqrt{\lambda} \quad \left(\lambda \simeq 1.56\,T^2\right), \tag{1.62}$$

$$c = \frac{\omega}{k}\left(=\frac{\lambda}{T}\right) = \frac{g}{\omega} \simeq 1.56T, \tag{1.63}$$

$$\Phi(x,y,t) = \mathrm{Re}\left[\,\phi(x,y)\,e^{i\omega t}\,\right], \quad \phi(x,y) = \frac{ga}{i\omega}\,e^{-Ky-iKx}. \tag{1.64}$$

1.5.4 Group velocity

In this section, we consider a narrow band of the component waves, with nearly equal wavelength and direction. A characteristic of the resulting distribution is that the waves travel in a group. The propagation velocity of the group of these waves is not the phase velocity but the group velocity, which is important in understanding the propagation of the wave energy, as will be explained later. However, as a complementary explanation for the group velocity, we consider here a purely kinematic analysis for the group of waves formed by two nearly equal plane waves.

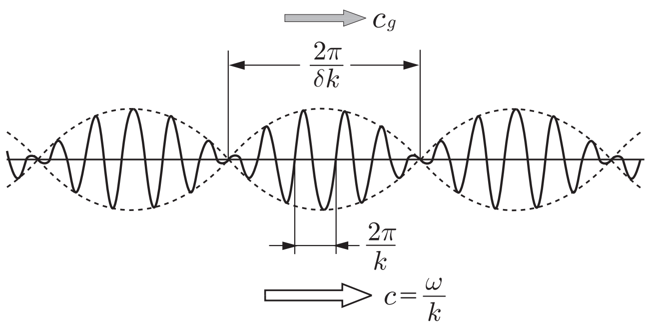


Fig.1.3 The amplitude modulation is represented by an envelope encompassing fundamental carrier waves and its propagation velocity (group velocity) is given by $\delta\omega/\delta k$.

Denoting the difference in the frequency and wavenumber of two nearly equal waves as $\delta\omega = \omega_2 - \omega_1$ and $\delta k = k_2 - k_1$, we can write the resulting wave profile as follows:

$$\begin{aligned} \eta &= \mathrm{Re}\left[A_1\, e^{i(\omega_1 t - k_1 x)} + A_2\, e^{i(\omega_2 t - k_2 x)} \right] \\ &= \mathrm{Re}\left[A_1\, e^{i(\omega_1 t - k_1 x)} \left\{ 1 + \frac{A_2}{A_1}\, e^{i(\delta\omega t - \delta k x)} \right\} \right]. \end{aligned} \tag{1.65}$$

Here the factor in braces represents an amplitude modulation, which will be slowly varying in both space and time, because both δk and $\delta\omega$ are assumed small. Substituting $A_2 = A_1 + \delta A$, taking the real part, and neglecting higher-order terms in δk, $\delta\omega$, and δA, we can obtain the following expression as the first-order approximation:

$$\eta = 2A_1 \cos\left[\frac{1}{2}\big(\delta\omega{\cdot}t - \delta k{\cdot}x\big) \right] \cos\big(\omega_1 t - k_1 x\big). \tag{1.66}$$

This type of wave motion is illustrated in Fig. 1.3.

The fundamental wave component is represented by the last cosine term with ω_1 and k_1, which is called the carrier wave. The amplitude is slowly varying and its envelope encompasses a group of carrier waves. As obvious from (1.66), the wavelength and period of the slowly varying part (the amplitude modulation) are $4\pi/\delta k$ and $4\pi/\delta\omega$, respectively; thus the length of one group of waves is $2\pi/\delta k$. The group velocity c_g, which is the propagation velocity of the group of waves represented by an envelope, is given by

$$c_g = \frac{\delta\omega}{\delta k}. \tag{1.67}$$

We consider the limiting case where $\delta\omega \to 0$ and $\delta k \to 0$, but at the same time both t and x are large enough so that the products $\delta\omega{\cdot}t$ and $\delta k{\cdot}x$ are finite. In this case, the amplitude modulation will persist and the group velocity (1.67) will approach the finite limit:

$$c_g = \frac{d\omega}{dk} = \frac{d(kc)}{dk} = c + k\frac{dc}{dk} = c - \lambda\frac{dc}{d\lambda}. \tag{1.68}$$

In general, the group and phase velocities differ, unless the phase velocity is independent of the wavelength, as can be seen from (1.68). As already studied, longer waves propagate faster and thus $dc/d\lambda > 0$ except for the shallow-water limit. Thus we can see that $c_g < c$ in general.

Let us work out the differentiation of (1.68) using the dispersion relation of (1.51). It may be convenient to differentiate after taking the logarithm of the dispersion relation. The result will be of the form

$$\frac{2}{\omega}\frac{d\omega}{dk_0} = \frac{1}{k_0} + \frac{2h}{\sinh 2k_0 h}.$$

Therefore,

$$c_g = \frac{1}{2}\frac{\omega}{k_0}\left[1 + \frac{2k_0 h}{\sinh 2k_0 h}\right] = \frac{c}{2}\left[1 + \frac{2k_0 h}{\sinh 2k_0 h}\right], \tag{1.69}$$

where $c = \omega/k_0$ is the phase velocity.

It is also obvious from (1.69) that $c_g < c$ in general except for a special case of $k_0 h \to 0$. In both limiting cases of deep water ($kh \gg 1$) and shallow water ($kh \ll 1$), (1.69) reduces to

$$c_g = \frac{1}{2}c \quad (\, kh \to \infty\,), \tag{1.70}$$

$$c_g = c \qquad (\, kh \to 0\,). \tag{1.71}$$

Namely, in deep water, the group velocity is precisely half of the phase velocity, and in the shallow-water limit, the group and phase velocities are the same.

1.6 Wave Energy and Its Propagation

By using the results in the preceding section, let us compute the energy density of the plane progressive waves. As an appropriate volume of fluid, we consider a vertical column, extending throughout the depth of the fluid and bounded above by the free surface. Then the energy density E, per unit area of the mean free surface above this column, can be computed from

$$E = \rho \int_{\eta}^{h} \left\{ \frac{1}{2}\nabla\Phi\cdot\nabla\Phi - gy \right\} dy, \tag{1.72}$$

where η denotes the free-surface elevation to be computed from

$$\eta = \frac{1}{g}\frac{\partial\Phi}{\partial t}\bigg|_{y=0}. \tag{1.73}$$

Thus we can see that $\eta = O(\Phi)$. In evaluating (1.72), contributions up to the 2nd order in Φ will be retained consistently and other higher-order terms will be discarded. Furthermore, the potential energy of the fluid below the still water surface will be omitted, because this is unrelated to the wave motion.

With these taken into account, (1.72) can be written as

$$E = \frac{1}{2}\rho\int_0^h \left\{ \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 \right\} dy + \frac{1}{2}\rho g\eta^2 + O(\Phi^3). \tag{1.74}$$

Then this equation will be averaged with respect to time over one cycle of the wave motion. For the calculation of the time average, it is convenient to use the following formula:

$$\left.\begin{aligned} \mathcal{F} &\equiv \mathrm{Re}\big[\,A\,e^{i\omega t}\big]\,\mathrm{Re}\big[\,B\,e^{i\omega t}\big] \\ \overline{\mathcal{F}} &\equiv \frac{1}{T}\int_0^T \mathcal{F}\,dt = \frac{1}{4}\big(AB^* + A^*B\big) = \frac{1}{2}\mathrm{Re}\big(\,A\,B^*\big) \end{aligned}\right\} \tag{1.75}$$

The velocity potential Φ for a plane progressive wave is given by (1.57) and (1.58). Thus by applying (1.75) to (1.74), the time average of the energy density (1.74) can be evaluated as follows:

$$\begin{aligned}\overline{E} &= \frac{1}{4}\,\rho\,\mathrm{Re}\int_0^h \left\{ \frac{\partial \phi}{\partial x}\frac{\partial \phi^*}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \phi^*}{\partial y} \right\} dy + \frac{1}{4}\,\rho g a^2 \\ &= \frac{1}{4}\,\rho\,(a\omega)^2\,\frac{1}{\sinh^2 k_0 h}\int_0^h \cosh 2k_0(y-h)\,dy + \frac{1}{4}\,\rho g a^2 \\ &= \frac{1}{4}\,\rho g a^2 + \frac{1}{4}\,\rho g a^2 = \frac{1}{2}\,\rho g a^2. \end{aligned} \tag{1.76}$$

We can see from this result that the kinetic and potential energies are the same and the density of the total wave energy is $\frac{1}{2}\rho g a^2$, which has nothing to do with water depth, wavenumber, and frequency.

Next, let us consider the rate of change with respect to time of the wave-energy density of a plane progressive wave. The calculation formula is already given by (1.37), but the first term on the right-hand side must be omitted because of no body in the present case. The control surface can be taken as the two vertical surfaces in the x-direction of a vertical column, separated with small distance δx, and these two vertical surfaces are denoted as $S(x)$ and $S(x+\delta x)$. The width of a vertical column in the direction parallel to the crest line of the wave is taken as unity. Then the rate of change of the wave energy within this small vertical column can be evaluated from

$$\begin{aligned}\frac{\partial E}{\partial t}\,\delta x &= \rho\left(\int_{S(x+\delta x)} - \int_{S(x)} \right) \frac{\partial \Phi}{\partial t}\frac{\partial \Phi}{\partial x}\,dy \\ &= \rho\,\frac{\partial}{\partial x}\left[\int_0^h \frac{\partial \Phi}{\partial t}\frac{\partial \Phi}{\partial x}\,dy \right]\delta x + O\big(\Phi^3\big). \end{aligned} \tag{1.77}$$

Using the complex representation (1.57) and (1.58) for the velocity potential and the formula (1.75) for the computation of time average, we obtain the following result:

$$\begin{aligned}\frac{\partial \overline{E}}{\partial t} &= \rho\frac{\partial}{\partial x}\left[\frac{1}{2}\,\mathrm{Re}\int_0^h \big(i\,\omega\phi\big)\frac{\partial \phi^*}{\partial x}\,dy \right] \\ &= -\frac{\partial}{\partial x}\left[\frac{1}{2}\rho(ga)^2\frac{k_0}{\omega}\frac{1}{\cosh^2 k_0 h}\int_0^h \frac{1+\cosh 2k_0(y-h)}{2}\,dy \right] \\ &= -\frac{\partial}{\partial x}\left[\frac{1}{2}\,\rho g a^2\,\frac{1}{2}\frac{\omega}{k_0}\left\{ 1 + \frac{2k_0 h}{\sinh 2k_0 h} \right\} \right]. \end{aligned} \tag{1.78}$$

Here we note that the quantity in brackets is the product of the energy density $\overline{E}$ given in (1.76) and the group velocity c_g given in (1.69). Therefore we may write (1.78) in the form

$$\frac{\partial \overline{E}}{\partial t} + c_g\,\frac{\partial \overline{E}}{\partial x} = 0. \tag{1.79}$$

This is written in a form of the advection equation with the group velocity as the transportation velocity. Therefore we can understand that the time-averaged energy density of a plane progressive wave will be transported with the group velocity in the same direction as that of the wave propagation.

2. Free-Surface Green Function

The velocity potential describing the flow around a floating or submerged body near the free surface can be expressed in a form of boundary integral equation by means of the Green's theorem. The free-surface Green function is a kernel function of the boundary integral equation, which is thus of great importance. Physically this Green function is the velocity potential of a periodic source with unit strength, satisfying all of the homogeneous boundary conditions except for a condition on the body surface. This chapter explains the details of the derivation of the free-surface Green function for a simpler case; that is, the 2-D problem in deep water.

2.1 Velocity Potential of Periodic Source with Unit Strength

The velocity potential for a plane progressive wave has been already explained and given as (1.56). This wave should be understood in a way that a disturbance (like an oscillating body or a wavemaker) exists somewhere at a large distance and only the progressive-wave part arrives at an observation point as a plane progressive wave. In other words, the fluid flow near the source of disturbance may be complicated, including the local waves, which decay with increasing the distance from the disturbance, in addition to the progressive wave. We shall consider an exact expression for this complicated flow induced by a periodic hydrodynamic source with unit strength. For simplicity, only the 2-D problem in water of infinite depth is considered here, but extension to more general or complex problems may be possible with the knowledge to be obtained in this chapter. Writing the time-dependent part as $e^{i\omega t}$, we shall obtain the velocity potential in the form

$$\Phi(x,y,t) = \mathrm{Re}\big[\, G(x,y)\, e^{i\omega t} \,\big] \tag{2.1}$$

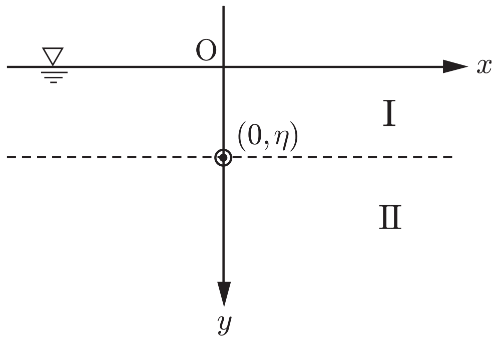


Fig. 2.1 Coordinate system

The unit source is assumed to be placed at $(0,\eta)$. Note that the induced flow is symmetric with respect to x, and thus we may assume x in subsequent analyses to be positive ($x>0$) and in the final result, x can be replaced with $|x|$ or more generally $|x-\xi|$ if the position of source in the x-axis is $x=\xi$.

The spatial part of the velocity potential (which may be called the Green function) must satisfy the following equations:

$$[L] \qquad \nabla^2 G = \delta(x)\,\delta(y-\eta) \tag{2.2}$$

$$[F] \qquad \frac{\partial G}{\partial y} + KG = 0 \qquad \text{on } y=0\ ; \quad K = \frac{\omega^2}{g} \tag{2.3}$$

$$[B] \qquad \frac{\partial G}{\partial y} = 0 \qquad \text{as } y \to \infty \tag{2.4}$$

$$[R] \qquad \text{Radiation condition; generated waves must be outgoing.} \tag{2.5}$$

It should be noted that the right-hand side of (2.2) is not zero but the Dirac's delta function with unit magnitude, which is related to the amount of flow out of the source singularity located at $x=0$ and

$y = \eta$. The radiation condition is not written explicitly here with a mathematical equation, which must be physically relevant and thus the waves generated must be outgoing from the source point. Since the time-dependent part is written as $e^{i\omega t}$, the solution satisfying the radiation condition must take a form of $G \sim A\, e^{-iKx}$ at $x \to \infty$ (in more general, $G \sim A\, e^{-iK|x|}$ at $|x| \to \infty$) on the free surface.

To seek the solution of (2.2)–(2.5), we will use the Fourier transform:

$$\left.\begin{aligned} G^*(k; y) &= \int_{-\infty}^{\infty} G(x, y)\, e^{-ikx}\, dx \\ G(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(k; y)\, e^{ikx}\, dk \end{aligned}\right\} \tag{2.6}$$

Utilizing the following relations

$$\int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = 1, \tag{2.7}$$

$$\int_{-\infty}^{\infty} \frac{d\, f(x)}{dx}\, e^{-ikx}\, dx = ik\, f^*(k), \tag{2.8}$$

we obtain the Fourier transforms of (2.2)–(2.4) as follows:

$$\frac{d^2 G^*}{dy^2} - k^2\, G^* = \delta(y - \eta) \tag{2.9}$$

$$\frac{dG^*}{dy} + KG^* = 0 \qquad \text{on } y = 0 \tag{2.10}$$

$$\frac{dG^*}{dy} = 0 \qquad \text{as } y \to \infty \tag{2.11}$$

Equation (2.9) is an ordinary differential equation with respect to y, subject to the boundary conditions of (2.10) and (2.11).

In order to obtain a solution taking account of singular behavior at $y = \eta$ due to the delta function on the right-hand side of (2.9), we shall consider the analysis according to the following procedure:

1) To avoid a singularity at $y = \eta$ for a moment, the fluid region is divided into the upper (Region I) and lower (Region II) parts, separated at $y = \eta$ as shown in Fig. 2.1. The homogeneous solutions valid in Region I and Region II are denoted as G_1^* and G_2^*, respectively.

2) Since (2.9) is a quadratic differential equation and the right-hand side can be zero except at $y = \eta$, both G_1^* and G_2^* includes two unknowns in general.

3) G_1^* must satisfy the free-surface boundary condition on $y = 0$, and G_2^* must satisfy the condition of no disturbance as $y \to \infty$. With these requirements, each of G_1^* and G_2^* includes one unknown (thus still two unknowns in total).

4) Two additional conditions which can be used to determine the remaining two unknowns may be provided by considering the matching conditions between G_1^* and G_2^* at $y = \eta$. In order to consider those conditions, let us integrate (2.9) over a small region crossing $y = \eta$; i.e. $\eta - \epsilon \leq y \leq \eta + \epsilon$ (where ϵ is assumed very small). The result of integration is

$$\left[\frac{dG^*}{dy} \right]_{\eta-\epsilon}^{\eta+\epsilon} - k^2 \int_{\eta-\epsilon}^{\eta+\epsilon} G^*\, dy = \int_{\eta-\epsilon}^{\eta+\epsilon} \delta(y - \eta)\, dy = 1. \tag{2.12}$$

We can see that this relation can be satisfied, provided that

$$G_2^*(\eta) = G_1^*(\eta), \tag{2.13}$$

$$\left.\frac{dG_2^*}{dy}\right|_{y=\eta} - \left.\frac{dG_1^*}{dy}\right|_{y=\eta} = 1. \tag{2.14}$$

These two matching conditions determine both G_1^* and G_2^* completely, thus we can obtain a unique solution $G^*(k,y)$ valid in the entire fluid region. Considering the inverse Fourier transform of thereby obtained solution in the Fourier-transformed domain may provide the desired solution in the physical domain of the velocity potential of a periodic source with unit strength; that is, the **Green function**.

Here we note that the Green function is defined as a function satisfying homogeneous boundary conditions like (2.3)–(2.5) and possessing a singularity like (2.2) at a particular point in the fluid region.

Let us obtain the Green function by the procedure described above. First a general homogeneous solution of (2.9) is given by

$$G^*(k;y) = C_1\, e^{|k|y} + C_2\, e^{-|k|y}. \tag{2.15}$$

Then G_1^* satisfying (2.10) and G_2^* satisfying (2.11) can be easily obtained and expressed in the form

$$G_1^*(k;y) = C\left\{ e^{|k|y} - e^{-|k|y} + \frac{2|k|}{|k|-K}\, e^{-|k|y} \right\}, \tag{2.16}$$

$$G_2^*(k;y) = D\, e^{-|k|y}, \tag{2.17}$$

where C and D are unknowns; which can be determined from (2.13) and (2.14) and hence the solutions of G_1^* and G_2^*. The obtained results may be expressed in a unified form as follows:

$$G^*(k;y) = -\frac{1}{2|k|}\left\{ e^{-|k||y-\eta|} - e^{-|k|(y+\eta)} \right\} - \frac{e^{-|k|(y+\eta)}}{|k|-K}. \tag{2.18}$$

The inverse Fourier transform of the above result can be written as

$$\begin{aligned} G(x,y;0,\eta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(k;y)\, e^{ikx}\, dk \\ &= -\frac{1}{2\pi}\int_0^{\infty} \left\{ e^{-k|y-\eta|} - e^{-k(y+\eta)} \right\} \frac{\cos kx}{k}\, dk - \frac{1}{\pi}\int_0^{\infty} \frac{e^{-k(y+\eta)}\cos kx}{k-K}\, dk. \end{aligned} \tag{2.19}$$

Here the first term on the right-hand side of (2.19) can be integrated analytically, with the result

$$-\int_0^{\infty} \left\{ e^{-k|y-\eta|} - e^{-k(y+\eta)} \right\} \frac{\cos kx}{k}\, dk = \log\frac{r}{r_1}, \tag{2.20}$$

where

$$\left.\begin{matrix} r \\ r_1 \end{matrix}\right\} = \sqrt{x^2 + (y \mp \eta)^2}.$$

Therefore the Green function in the physical domain satisfying the free-surface and bottom boundary conditions are expressed in the form

$$G(x,y;0,\eta) = \frac{1}{2\pi}\big(\log r - \log r_1\big) - \frac{1}{\pi}\int_0^{\infty} \frac{e^{-k(y+\eta)}\cos kx}{k-K}\, dk. \tag{2.21}$$

The first term on the right-hand side; i.e. $\frac{1}{2\pi}\log r$ represents the velocity potential of a source with unit strength in an unbounded fluid. As explained before, this term has the delta-function singularity on the right-hand side of (2.2) and this term is referred to as the fundamental (or principal) solution in the theory of partial differential equations. The other terms on the right-hand side of (2.21) represent the free-surface effect and these are regular in the entire fluid domain under consideration.

Up to this point in our derivation, the radiation condition has not been considered explicitly. Let us consider how the radiation condition will be satisfied through the treatment of the integral appearing in (2.21). This integral is a singular integral, with its integrand becoming singular at $k = K$. Mathematically, there can be three different ways in treating this kind of singular integrals. Namely,

$$\int_0^\infty \frac{e^{-ky}\cos kx}{k-K}\,dk = \begin{cases} \displaystyle\lim_{\mu\to 0}\int_0^\infty \frac{e^{-ky}\cos kx}{k-(K-i\mu)}\,dk \equiv I_1 \\ \displaystyle\lim_{\mu\to 0}\int_0^\infty \frac{e^{-ky}\cos kx}{k-(K+i\mu)}\,dk \equiv I_2 \\ \displaystyle\frac{1}{2}\left(I_1+I_2\right) \equiv I_3 \end{cases} \tag{2.22}$$

Although these three are mathematically correct, the results to be obtained may be physically different. Among these, we should select a physically relevant result, which satisfies the radiation condition and thus must have a form of $A\,e^{-iK|x|}$ as $|x|\to\infty$. Through this kind of selection in the treatment of the singular integral in (2.21), we can satisfy the radiation condition.

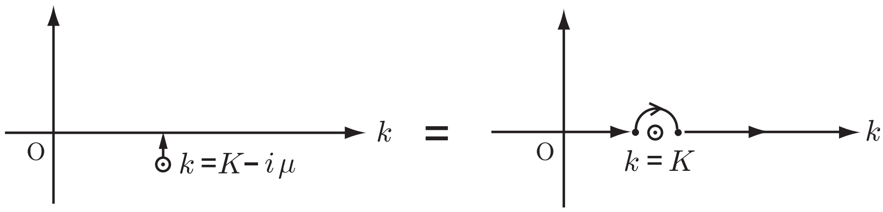


Fig.2.2 A treatment of singular point on the real axis.

First, let us consider the treatment of I_1 in (2.22). Due to the existence of a small negative imaginary part in the wavenumber $(K-i\mu)$ and a limit of $\mu\to 0$, we may deform the integration path on the real axis as shown in Fig. 2.2. Then with the residue theorem, the following relation holds:

$$I_1 = \lim_{\mu\to 0}\int_0^\infty \frac{e^{-ky}\cos kx}{k-(K-i\mu)}\,dk = \oint_0^\infty \frac{e^{-ky}\cos kx}{k-K}\,dk - \pi i\, e^{-Ky}\cos Kx. \tag{2.23}$$

The first integral on the right-hand side must be treated as Cauchy's principal-value integral; that is, the integration range excludes a very small neighborhood of $k=K$ as explicitly shown below

$$L_C \equiv \oint_0^\infty \frac{e^{-ky}\cos kx}{k-K}dk = \lim_{\epsilon\to 0}\left\{\int_0^{K-\epsilon} + \int_{K+\epsilon}^\infty\right\}\frac{e^{-ky}\cos kx}{k-K}\,dk. \tag{2.24}$$

It is noteworthy that this principal-value integral must be of real quantity.

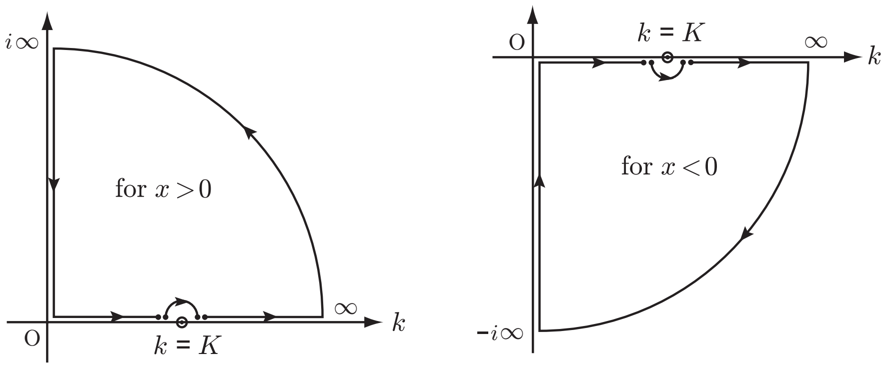


Fig.2.3 Transform of integration path in the complex plane.

Next, to make clear the physical meaning of this principal-value integral, let us transform it by using a complex-plane integral. For that purpose, we will consider the following integral in the complex plane

$$J \equiv \oint_C \frac{e^{-\zeta y + i\zeta x}}{\zeta - K}\, d\zeta. \tag{2.25}$$

The integration path in the complex plane must be taken such that the integrand at infinity does not diverge. To investigate the behavior at infinity, $\zeta = R\, e^{i\theta}$ $(R \to \infty)$ is substituted into the exponential function in (2.25). Then we have

$$e^{-\zeta y + i\zeta x} = e^{-R\{y\cos\theta + x\sin\theta\} - iR\{y\sin\theta - x\cos\theta\}}. \tag{2.26}$$

For $x > 0$, this function decays exponentially as $R \to \infty$, as long as θ is taken within $0 \le \theta \le \pi/2$. (Note that $y \ge 0$ in the fluid domain.) With this information, we take the integration path in the first quadrant as shown on the left in Fig. 2.3. Needless to say, for the case of $x < 0$, the path must be taken in the fourth quadrant, i.e. $-\pi/2 \le \theta \le 0$ as shown on the right in Fig. 2.3. Even in this case, the result to be obtained will be the same as that for $x > 0$, if x is replaced with $|x|$ in the final result. (Physically this is natural, because the flow induced by a source must be symmetric with respect to x.)

Once an appropriate integration path is taken in the complex plane depending on the sign of x, the contribution from a path along the quadrant at infinity is zero. Since there are no singular points inside the round integration path shown in Fig. 2.3, integral J defined by (2.25) must be zero on account of Cauchy's fundamental theorem. Writing each contribution from the round integration path in Fig. 2.3, we can write the final result as follows:

$$J = \oint_0^\infty \frac{e^{-ky+ikx}}{k-K}\, dk - \pi i\, e^{-Ky+iKx} + \int_\infty^0 \frac{e^{-iky-kx}}{ik-K}\, i\, dk = 0. \tag{2.27}$$

Thus
$$\oint_0^\infty \frac{e^{-ky+ikx}}{k-K}\, dk = \int_0^\infty \frac{e^{-iky-kx}}{k+iK}\, dk + \pi i\, e^{-Ky+iKx}. \tag{2.28}$$

The integration on the left-hand side is obtained from the integral on the real axis except at $k = K$ and thus associated with Cauchy's principal-value integral. Therefore, by taking only the real part of (2.28), we can see that Cauchy's principal-value integral L_C defined in (2.24) can be transformed in the following form

$$\begin{aligned} L_C = \oint_0^\infty \frac{e^{-ky}\cos kx}{k-K}\, dk &= \mathrm{Re} \oint_0^\infty \frac{e^{-ky+ikx}}{k-K}\, dk \\ &= \mathrm{Re} \int_0^\infty \frac{(k-iK)\, e^{-iky}}{(k-iK)(k+iK)}\, e^{-kx}\, dk - \pi\, e^{-Ky}\sin Kx \\ &= \int_0^\infty \frac{k\cos ky - K\sin ky}{k^2+K^2} e^{-kx} dk - \pi\, e^{-Ky}\sin Kx. \end{aligned} \tag{2.29}$$

We note again that L_C is of real quantity, despite that we have used a technique of complex integral.

Substituting this result as the first integral on the right-hand side of (2.23) and replacing x with $|x|$, we can obtain another expression for I_1 defined in (2.22) in the form

$$\begin{aligned} I_1 &= L_C - \pi i\, e^{-Ky}\cos Kx \\ &= \int_0^\infty \frac{k\cos ky - K\sin ky}{k^2+K^2}\, e^{-k|x|} dk - \pi i\, e^{-Ky - iK|x|}. \end{aligned} \tag{2.30}$$

This expression enables us to understand the physical meaning of each term. As is clear by considering the case of $|x| \to \infty$, the first term on the right-hand side in (2.30) represents a local wave which exists only near the source and decays rapidly as the distance from the source increases. The second term given by a complex exponential function represents a progressive wave which propagates away from the source and thus represents physically expected result. Therefore, we can conclude that the treatment of I_1 for the singular integral defined in (2.22) does satisfy the radiation condition.

Details of the transformation according to I_2 in (2.22) will be omitted here, but it may be easily understood from the difference between I_1 and I_2 that the final result to be obtained from I_2 is just the complex conjugate of the result obtained for I_1. Therefore the result of I_2 represents an incoming progressive wave, which is physically inappropriate and thus must be discarded. The treatment of I_3 in (2.22) is the average of I_1 and I_2 with equal magnitude, providing not progressive but stationary wave which is also physically inappropriate.

Summarizing correct expressions of the free-surface Green function satisfying the radiation condition of outgoing waves away from a point source, we have the following results:

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\log\frac{r}{r_1} - \frac{1}{\pi}\lim_{\mu\to 0}\int_0^\infty \frac{e^{-k(y+\eta)}\cos k(x-\xi)}{k-(K-i\mu)}\,dk \tag{2.31}$$

$$= \frac{1}{2\pi}\log\frac{r}{r_1} - \frac{1}{\pi}\oint_0^\infty \frac{e^{-k(y+\eta)}\cos k(x-\xi)}{k-K}\,dk + i\,e^{-K(y+\eta)}\cos K(x-\xi) \tag{2.32}$$

$$= \frac{1}{2\pi}\log\frac{r}{r_1} - \frac{1}{\pi}\int_0^\infty \frac{k\cos k(y+\eta) - K\sin k(y+\eta)}{k^2+K^2}\,e^{-k|x-\xi|}\,dk + i\,e^{-K(y+\eta)-iK|x-\xi|}. \tag{2.33}$$

As we can envisage from the transformation above, a key point to satisfy the correct radiation condition is to regard the wavenumber K as not real but complex quantity with very small negative imaginary part $(K-i\mu,\ \mu>0)$. (However, we note that μ must tend to zero in the end of analysis after transformation of singular integrals.)

Next, let us consider a physical meaning of this small value of μ. In the problem treated here, the unsteady motion of fluid is assumed to be harmonic in time with circular frequency ω, which actually makes it difficult a little to satisfy the radiation condition, as we have seen in the analysis shown above. In reality, the unsteady motion may start from an initial state of rest and become steady state of harmonic oscillation after going through a transient stage. Thus, if the problem is treated as an initial-value problem, there must be no ambiguity in the solution, representing physically relevant phenomenon of outgoing waves away from a disturbance. This situation in reality may be mimicked in an approximate way simply by making the circular frequency ω slightly complex, with negative imaginary part, so that the flow induced vanishes for $t\to-\infty$. Namely, the time-dependent part $e^{i\omega t}$ must be modified in the form $e^{\epsilon t}\,e^{i\omega t} = e^{i(\omega-i\epsilon)t}$ and thus we should write (2.1) as

$$\Phi(x,y,t) = \mathrm{Re}\Big[\,G(x,y)\,e^{i(\omega-i\epsilon)t}\Big]. \tag{2.34}$$

Starting with this expression using $\omega-i\epsilon$ instead of ω, the wavenumber $K=\omega^2/g$ appearing in the free-surface boundary condition must be modified as follows:

$$K = \frac{(\omega-i\epsilon)^2}{g} \simeq \frac{\omega^2}{g} - i\mu \quad \Big(\mu = 2\frac{\omega\epsilon}{g}\Big). \tag{2.35}$$

This is equivalent to the treatment of I_1 defined in (2.22), and thus we can obtain automatically the correct expression of the Green function, without being annoyed with satisfaction of the radiation condition.

Going back to the free-surface condition (2.3) with understanding of (2.34) and (2.35), we may write the free-surface condition in the form

$$\frac{\partial G}{\partial y} + \big(K - i\mu\big)G = 0 \quad \text{on } y = 0. \tag{2.36}$$

This can be regarded as a combined expression of the free-surface and radiation conditions. In fact, by following the solution procedure explained up to (2.21) in terms of (2.36) instead of (2.3), we can easily confirm that the expression corresponding to (2.21) takes the form

$$G(x,y;0,\eta) = \frac{1}{2\pi}\big(\log r - \log r_1\big) - \frac{1}{\pi}\int_0^\infty \frac{e^{-k(y+\eta)}\cos kx}{k-(K-i\mu)}\,dk. \tag{2.37}$$

This is completely the same as (2.31). Therefore by following the transformation explained as the treatment of I_1, we can obtain the correct result satisfying the radiation condition, without recourse to the argument on whether the obtained result is physically plausible.

2.2 Green's Theorem

Now that we have studied the Green function as the velocity potential of the flow generated by a periodically oscillating source, we shall derive the velocity potential of the flow induced by a general-shaped floating body by means of the Green function. Explanation here starts with Gauss' divergence theorem in a 2-D fashion:

$$\iint_V \nabla\cdot\boldsymbol{A}\,dS = -\oint_S \boldsymbol{n}\cdot\boldsymbol{A}\,ds. \tag{2.38}$$

Here $\boldsymbol{n}$ denotes the normal vector, which is defined positive when pointing from the boundary into the fluid domain, resulting in the minus sign on the right-hand side of (2.38).

Let us consider $\boldsymbol{A} = \phi\nabla G$, with ϕ the velocity potential to be obtained for a general-shaped body and G the Green function obtained in the preceding section. The result can be written as

$$\iint_V \big(\nabla\phi\cdot\nabla G + \phi\nabla^2 G\big)\,dS = -\oint_S \phi\,\frac{\partial G}{\partial n}\,ds. \tag{2.39}$$

Similarly by considering $\boldsymbol{A} = G\nabla\phi$, we have the following

$$\iint_V \big(\nabla G\cdot\nabla\phi + G\nabla^2\phi\big)\,dS = -\oint_S G\,\frac{\partial \phi}{\partial n}\,ds. \tag{2.40}$$

Subtracting (2.40) from (2.39), it follows that

$$\iint_V \big(\phi\nabla^2 G - G\nabla^2\phi\big)\,dS = \oint_S \left\{\frac{\partial\phi}{\partial n}\,G - \phi\,\frac{\partial G}{\partial n}\right\}ds. \tag{2.41}$$

Here we note that $\nabla^2\phi = 0$ throughout the fluid region but the Green function G has a singularity at the source point $(x,y) = (\xi,\eta)$ and thus satisfies

$$\nabla^2 G = \delta(x-\xi)\,\delta(y-\eta). \tag{2.42}$$

In subsequent derivation we shall perform the integration with respect to the coordinates of the source point (ξ,η); which may be allowed by noting the reciprocity property of the Green function:

$$G(x,y;\xi,\eta) = G(\xi,\eta;x,y). \tag{2.43}$$

Substituting (2.42) in (2.41), we can readily obtain an important result of the form

$$\phi(\mathrm{P}) = \oint_S \left\{\frac{\partial\phi(\mathrm{Q})}{\partial n_Q} - \phi(\mathrm{Q})\frac{\partial}{\partial n_Q}\right\}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}), \tag{2.44}$$

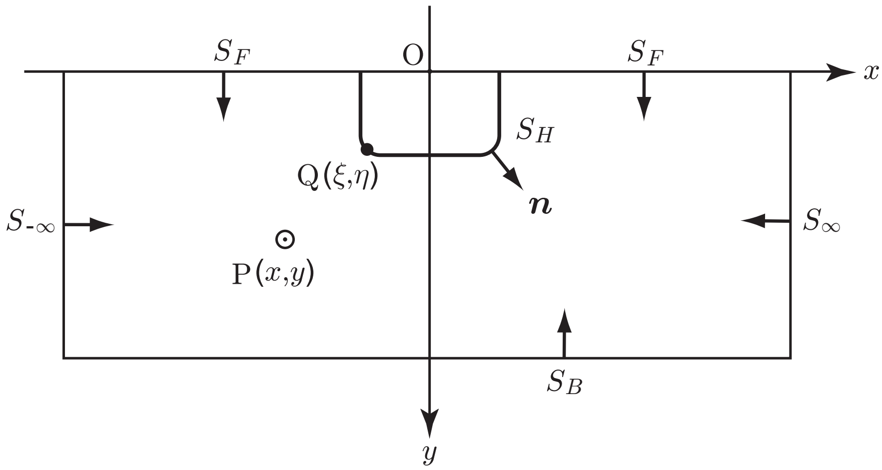


Fig.2.4 Coordinate system and notations.

where P $= (x, y)$ is the field point and Q $= (\xi, \eta)$ is the integration (or the source) point. This form of representation for the velocity potential is called Green's theorem.

It should be noted that the area of integration (S) in (2.44) is all the boundary surrounding the fluid under consideration, consisting of the hull surface of a floating body (S_H), the free surface (S_F), the bottom of water (S_B) and artificial vertical lines far from the body as the radiation boundary ($S_{\pm\infty}$), as shown in Fig. 2.4.

At this moment, let us examine the value of integrand of (2.44) on the boundary surface, for which we summarize here the boundary conditions satisfied by ϕ(Q) and G(P; Q). With (2.43) kept in mind, those can be expressed as

$$\left.\begin{array}{lll} [S_F] & \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\phi}{\partial\eta} = -K\phi\ ; & \dfrac{\partial G}{\partial n} = \dfrac{\partial G}{\partial\eta} = -KG \quad \text{on } \eta = 0 \\ [S_B] & \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\phi}{\partial\eta} = 0 \qquad ; & \dfrac{\partial G}{\partial n} = \dfrac{\partial G}{\partial\eta} = 0 \quad \text{as } \eta \to \infty \\ [S_{\pm\infty}] & \dfrac{\partial}{\partial n} = \mp\dfrac{\partial}{\partial\xi}\ , \quad \phi \sim A\, e^{-K\eta\mp iK\xi}\ ; & G \sim B\, e^{-K\eta\mp iK\xi} \end{array}\right\} \tag{2.45}$$

Substituting these into (2.44), we can see that the integrand of (2.44) becomes zero on S_F, S_B, and $S_{\pm\infty}$. As a result, the integral on the floating-body surface (S_H) remains only as nonzero contribution. Thus we have an expression for the velocity potential at the field point P(x, y) in the fluid as follows:

$$\phi(\mathrm{P}) = \int_{S_H} \left\{ \frac{\partial\phi(\mathrm{Q})}{\partial n_Q} - \phi(\mathrm{Q})\, \frac{\partial}{\partial n_Q} \right\} G(\mathrm{P}; \mathrm{Q})\, ds(\mathrm{Q}). \tag{2.46}$$

We should recall that the Green function was derived to satisfy all homogeneous boundary conditions which are the same as those to be satisfied by the velocity potential to be obtained for a general-shaped floating body. It may be clear from the argument above that the contribution becomes zero from the boundary where a homogeneous boundary condition is imposed and the Green function is obtained to satisfy the same homogeneous condition. This fact is greatly advantageous from the viewpoint of reducing the number of unknowns in numerical computations (because generally $\partial\phi/\partial n$ or ϕ is unknown on boundaries). In return for this advantage, the free-surface Green function becomes complicated as compared to the fundamental solution ($\log r$), as seen in (2.31)–(2.33), especially the integral term with respect to k representing the local wave appears to cause a problem. However, with recent computers, this integral term can be evaluated with great accuracy and less computation time.

Although the free-surface Green function $G(\mathrm{P};\mathrm{Q})$ can be evaluated without any problem and $\partial\phi(\mathrm{Q})/\partial n_Q$ can be given explicitly through an inhomogeneous boundary condition on the body surface, the velocity potential $\phi(\mathrm{Q})$ on the body surface is unknown. Therefore (2.46) is useless as it is. To determine the velocity potential on the body surface, we consider a limiting case where the field point $\mathrm{P}(x,y)$ is placed on the body surface. In this case, (2.46) may provide an integral equation for the velocity potential on the body surface, because both P and Q are on the body surface. However, there is one important thing to be noted. In this limit, as is clear from the argument on the amount of net flux from a point source, the amount of flux into the fluid region when $\mathrm{P}(x,y)$ is on the boundary must be just half of that when $\mathrm{P}(x,y)$ is in the fluid (if the body surface is smooth). That is to say, (2.42) in this case must be modified as

$$\nabla^2 G = \frac{1}{2}\,\delta(x-\xi)\,\delta(y-\eta). \tag{2.47}$$

Using this equation instead of (2.42) and following the same argument and transformation described before, we can obtain the following equation:

$$\frac{1}{2}\,\phi(\mathrm{P}) = \int_{S_H}\left\{\frac{\partial\phi(\mathrm{Q})}{\partial n_Q} - \phi(\mathrm{Q})\,\frac{\partial}{\partial n_Q}\right\}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}), \tag{2.48}$$

where $\mathrm{P}(x,y)$ is situated on S_H.

Equation (2.48) can be rewritten in a form of integral equation as follows:

$$\frac{1}{2}\,\phi(\mathrm{P}) + \int_{S_H}\phi(\mathrm{Q})\frac{\partial}{\partial n_Q}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}) = \int_{S_H}\frac{\partial\phi(\mathrm{Q})}{\partial n_Q}\,G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}). \tag{2.49}$$

Since $\partial\phi/\partial n$ can be given from the body boundary condition (its explicit form will be explained later), the integral on the right-hand side can be computed. Thus (2.49) can be regarded as an integral equation for the velocity potential on the body surface. The method of solving this kind of integral equation for the velocity potential on the body surface is referred to as the direct boundary element method or the free-surface Green function method.

Once the velocity potential $\phi(\mathrm{Q})$ on the body surface has been determined by solving (2.49) with an appropriate numerical method, we can compute the velocity potential $\phi(\mathrm{P})$ at any point in the fluid region in terms of (2.46).

2.3 Kochin Function

Let us consider the behavior of the velocity potential given by (2.46) far from a floating body and examine the difference in the asymptotic form of the velocity potentials (or equivalently wave elevations) for a point source and for a floating body.

The field point $\mathrm{P}(x,y)$ on the right-hand side of (2.46) appears only in the Green function $G(\mathrm{P};\mathrm{Q})$. Thus an asymptotic form of the velocity potential $\phi(\mathrm{P})$ as $|x|\to\infty$ can be obtained by substituting the asymptotic form of $G(\mathrm{P};\mathrm{Q})$; which can be easily obtained simply by discarding the local-wave terms in (2.33) and the result is expressed as

$$G(\mathrm{P};\mathrm{Q}) \sim i\,e^{-K(y+\eta)\mp iK(x-\xi)} = i\,e^{-K\eta\pm iK\xi}\,e^{-Ky\mp iKx} \quad \text{as } x\to\pm\infty. \tag{2.50}$$

Substituting this in (2.46), we can obtain the following result:

$$\phi(x,y) \sim i\,H^{\pm}(K)\,e^{-Ky\mp iKx} \quad \text{as } x\to\pm\infty, \tag{2.51}$$

where

$$H^{\pm}(K) = \int_{S_H}\left(\frac{\partial\phi}{\partial n} - \phi\,\frac{\partial}{\partial n}\right)e^{-K\eta\pm iK\xi}\,ds(\xi,\eta). \tag{2.52}$$

In the linear theory, the wave elevation on the free surface can be computed in terms of the velocity potential on $y = 0$, and thus the asymptotic form of the wave (which must be a progressive wave) can be computed in terms of (2.51) in the form

$$\left\{ \begin{aligned} \zeta(x,t) &= \mathrm{Re}\left[\,\zeta(x)\, e^{i\omega t}\,\right], \\ \zeta(x) &= \frac{i\omega}{g}\,\phi(x,0) \sim -\frac{\omega}{g}\,H^{\pm}(K)\,e^{\mp iKx} \quad \text{as } x \to \pm\infty. \end{aligned} \right. \tag{2.53}$$

We can see from (2.53) that $H^{\pm}(K)$ defined by (2.52) is equivalent to the complex amplitude (i.e. real amplitude and phase) of the progressive wave generated by a floating body. Thus this function is called the wave amplitude function or the Kochin function.

Comparing with (2.50) induced by a point source, we can see that characteristics of the wave outgoing from a disturbance with wavenumber K is of course the same. However the amplitude and phase are changed into a form of the Kochin function, which includes the effects of geometry and motion of a floating body, because $\partial\phi/\partial n$ in the definition of (2.52) is given by the body boundary condition as a function of body geometry and motion and thus the velocity potential ϕ as a solution of (2.49) is also a function of body geometry and motion.

3. Two-Dimensional Wave Making Theory

In order to understand the characteristics of wave-induced motions of a floating body and the theory of wave-energy absorption and perfect reflection, it is necessary to understand properly the so-called radiation and diffraction problems and various hydrodynamic relations satisfied between the waves generated by the body and hydrodynamic forces acting on the body.

3.1 Boundary Condition on a Floating Body

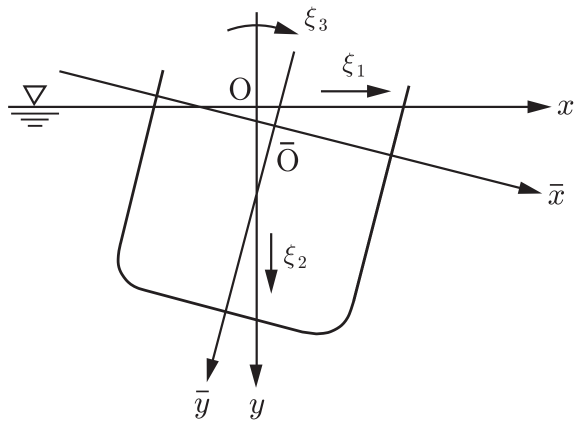


Fig. 3.1 Coordinate system and notations.

As already explained in Section 1.3, the boundary condition to be satisfied on an oscillating body can be obtained by considering the kinematic condition; that is, the zero substantial derivative of a function describing the body surface. However, the substantial derivative is defined in a space-fixed coordinate system, whereas the body geometry is normally defined as time invariable with a body-fixed coordinate system. Thus the difference between these coordinate systems associated with oscillation of a body must be taken into account in evaluating the substantial derivative.

As shown in Fig. 3.1, let the space-fixed coordinate system be denoted as O-xy and the body-fixed coordinate system be denoted as $\bar{O}$-$\bar{x}\bar{y}$. With assumption of small amplitude of body motions, the relation between the two coordinate systems can be written as $\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{\alpha}(t)$, where $\boldsymbol{x} = (x, y)$ and $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})$ are position vectors, and $\boldsymbol{\alpha}(t)$ denotes the displacement vector of body motion; which may be expressed as

$$\left.\begin{aligned} \boldsymbol{\alpha}(t) &= \boldsymbol{i}\,\xi_1(t) + \boldsymbol{j}\,\xi_2(t) + \boldsymbol{k}\,\xi_3(t) \times \boldsymbol{x} \\ \xi_j(t) &= \mathrm{Re}\left\{ X_j\, e^{i\omega t} \right\} \end{aligned}\right\} \tag{3.1}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit vectors along the x-, y-, and z-axes in the space-fixed coordinate system, and j =1, 2, 3 in subscript denote sway, heave, and roll respectively, with complex amplitude X_j in the j-th mode of motion.

The geometry of body surface is time-independent with the body-fixed coordinates $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})$, and thus in terms of $\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{\alpha}(t)$, the body surface can be described generally as follows:

$$F(\bar{\boldsymbol{x}}) = F(\boldsymbol{x} - \boldsymbol{\alpha}(t)) = 0. \tag{3.2}$$

Therefore the substantial derivative of (3.2) takes the form

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla\Phi \cdot \nabla F = -\dot{\boldsymbol{\alpha}}(t) \cdot \nabla F + \nabla\Phi \cdot \nabla F = 0. \tag{3.3}$$

Dividing this equation with $|\nabla F|$ and noting that the normal vector can be defined as $\boldsymbol{n} = \nabla F / |\nabla F|$, (3.3) may be rewritten as

$$\nabla\Phi \cdot \boldsymbol{n} = \frac{\partial \Phi}{\partial n} = \dot{\boldsymbol{\alpha}}(t) \cdot \boldsymbol{n} \equiv V_n. \tag{3.4}$$

Exactly speaking, the normal vector $\boldsymbol{n}$ and associated normal derivative in (3.4) are expressed in the space-fixed reference frame. However, computing the normal vector is easier with body-fixed coordinates. The distinction between the space-fixed and body-fixed coordinate systems may be a source of second-order effects that can be neglected in the linear theory. With this understanding, we shall not distinguish $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})$ and $\boldsymbol{x} = (x, y)$ in what follows. Then, writing the time-dependent term as $e^{i\omega t}$ and the velocity potential as $\Phi(\boldsymbol{x}, t) = \mathrm{Re}\{\, \phi(\boldsymbol{x})\, e^{i\omega t} \}$, (3.4) can be expressed as follows:

$$\frac{\partial \phi}{\partial n} = \sum_{j=1}^{3} i\omega X_j \, n_j \tag{3.5}$$

where

$$\left.\begin{aligned} n_1 = n_x = \frac{\partial x}{\partial n}, \quad n_2 = n_y = \frac{\partial y}{\partial n} \\ n_3 = x\, n_2 - y\, n_1 = (\boldsymbol{r} \times \boldsymbol{n})_3 = \varepsilon_{3jk} x_j n_k \quad (x_1 = x,\ x_2 = y) \end{aligned}\right\} \tag{3.6}$$

Since the right-hand side of (3.5) is in a form of linear superposition of each mode of motion, it looks that we may write the velocity potential in a corresponding form, like

$$\phi(\boldsymbol{x}) = \sum_{j=1}^{3} \phi_j(\boldsymbol{x}) \equiv \sum_{j=1}^{3} i\omega X_j \, \varphi_j(\boldsymbol{x}).$$

However, we should note that the velocity potential $\phi(\boldsymbol{x})$ must include the incident-wave potential, say $\phi_0(\boldsymbol{x})$, as an input of wave-induced motions. Therefore, in order to satisfy the body-boundary condition (3.5), the velocity potential must include the scattering component, say $\phi_4(\boldsymbol{x})$, which represents the interaction of incident waves with the body and the normal derivative of the sum $\phi_0 + \phi_4$ must be equal to zero on the body surface. Summarizing the above, we may write the velocity potentials and the boundary conditions for each component as follows:

$$\phi(\boldsymbol{x}) = \sum_{j=0}^{4} \phi_j(\boldsymbol{x}) = \frac{ga}{i\omega}\Big\{ \varphi_0(\boldsymbol{x}) + \varphi_4(\boldsymbol{x}) \Big\} + \sum_{j=1}^{3} i\omega X_j \varphi_j(\boldsymbol{x}), \tag{3.7}$$

$$\frac{\partial}{\partial n}\big(\varphi_0 + \varphi_4 \big) = 0, \tag{3.8}$$

$$\frac{\partial \varphi_j}{\partial n} = n_j \quad (j = 1, 2, 3). \tag{3.9}$$

Here φ_0 denotes the incident-wave potential normalized with $ga/i\omega$ (where a and ω are the amplitude and circular frequency of the incident wave, respectively). For the case of infinite water depth and incoming from the positive x-axis, φ_0 is given explicitly as

$$\varphi_0(x, y) = e^{-Ky + iKx}. \tag{3.10}$$

As already mentioned, φ_4 represents the **scattering potential** introduced to satisfy the boundary condition (3.8), which has nothing to do with the body motions at all. Thus the physical situation in considering φ_4 is that the body is fixed in space (no body motions) and incident waves are diffracted by the presence of a body. The sum of $\varphi_0 + \varphi_4 \equiv \varphi_D$ is referred to as the **diffraction potential** in this lecture note. (Note that some authors may call φ_4 the diffraction potential.)

On the other hand, φ_j ($j = 1 \sim 3$) represents the flow induced by the j-th mode of motion in an otherwise calm water (without incident waves) and is referred to as the **radiation potential**, normalized with velocity $i\omega X_j$. Schematic illustration of the diffraction and radiation problems is shown in Fig. 3.2.

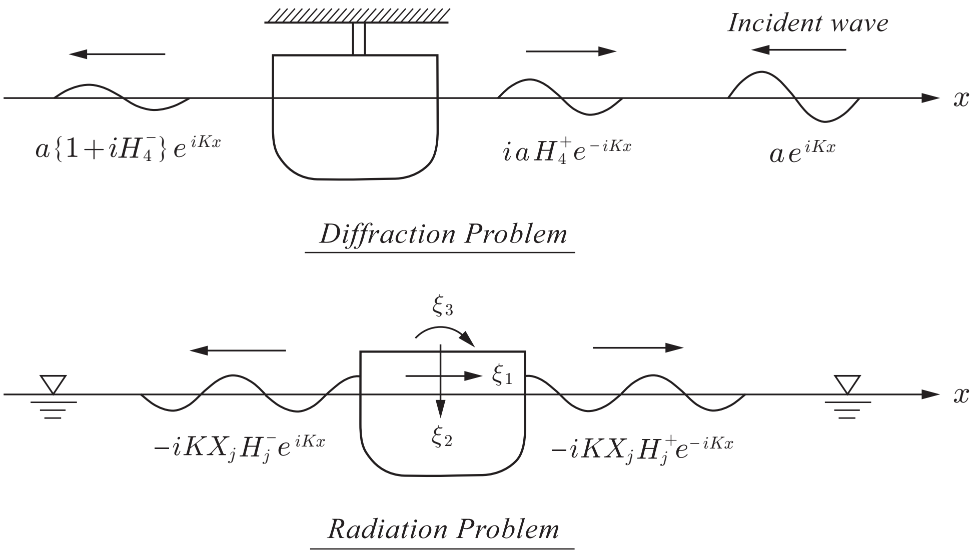


Fig.3.2 Schematic illustration for the diffraction and radiation problems.

3.2 Decomposition of Kochin Function and Progressive Waves

As described in the preceding section, within the framework of linear theory, a complicated real problem of a floating body oscillating in waves can be decomposed into the diffraction problem and the radiation problems for each mode of motions. Thus the progressive waves generated by the body (equivalently the Kochin function) can be decomposed in the same fashion; which will be described below.

The velocity potentials for the disturbance due to the presence of a body are given by (3.7) with incident-wave term ϕ_0 excluded. Thus, substituting those in the definition of the Kochin function (2.51), we can write it in the form of superposition of each component as follows:

$$H^{\pm}(K) = \frac{ga}{i\omega} H_4^{\pm}(K) + \sum_{j=1}^{3} i\omega X_j H_j^{\pm}(K) = \frac{ga}{i\omega} \left\{ H_4^{\pm}(K) - K \sum_{j=1}^{3} \frac{X_j}{a} H_j^{\pm}(K) \right\}. \tag{3.11}$$

Here the asymptotic form of each component of the disturbance potential can be written as follows:

$$\varphi_j(x, y) \sim i\, H_j^{\pm}(K)\, e^{-Ky \mp iKx} \quad \text{as } x \to \pm\infty, \tag{3.12}$$

$$H_j^{\pm}(K) = \int_{S_H} \left(\frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial}{\partial n} \right) e^{-K\eta \pm iK\xi}\, ds(\xi, \eta) \quad (j = 1 \sim 4). \tag{3.13}$$

It should be noted that $\partial\varphi_j/\partial n$ on the body surface is given by (3.9) as a real quantity but φ_j itself is complex even on the body surface (because the free-surface Green function in the integral equation for φ_j is complex), and consequently the Kochin function by (3.13) is generally complex.

In terms of the Kochin function, the complex amplitude of progressive wave generated by the body is expressed as

$$\begin{aligned} \zeta(x) &= -\frac{\omega}{g} \left\{ \frac{ga}{i\omega} H_4^{\pm}(K) + \sum_{j=1}^{3} i\omega X_j H_j^{\pm}(K) \right\} e^{\mp iKx} \\ &= a \left\{ iH_4^{\pm}(K) - iK \sum_{j=1}^{3} \frac{X_j}{a} H_j^{\pm}(K) \right\} e^{\mp iKx} \quad \text{as } x \to \pm\infty. \end{aligned} \tag{3.14}$$

This wave elevation can be rewritten in a decomposed form as follows:

$$\zeta(x,t) = \mathrm{Re}\Big[\,\zeta(x)\,e^{i\omega t}\,\Big] \equiv \mathrm{Re}\Bigg[\,\sum_{j=1}^{4}\zeta_j^{\pm}\,e^{i(\omega t \mp Kx)}\,\Bigg], \tag{3.15}$$

where

$$\zeta_j^{\pm} = -iKX_j H_j^{\pm}(K) \quad \text{for } j = 1 \sim 3:\ \text{radiation wave}, \tag{3.16}$$

$$\zeta_4^{\pm} = ia\,H_4^{\pm}(K) \qquad \text{for scattered wave}. \tag{3.17}$$

In the radiation problem, the wave amplitude of generated wave is usually expressed as the ratio to the amplitude of body motion. Specifically, when the body motion is given by $\xi_j(t) = \mathrm{Re}\{\,X_j\,e^{i\omega t}\}$, the elevation of progressive wave far from the body is written as $\mathrm{Re}\{\,X_j\bar{A}_j\,e^{i\varepsilon_j^{\pm}}\,e^{i(\omega t \mp Kx)}\}$, with $\bar{A}_j$ and $\varepsilon_j^{\pm}$ being the wave amplitude ratio and the phase difference, respectively. Comparing this expression with (3.16), we can obtain the following relation:

$$\bar{A}_j\,e^{i\varepsilon_j^{\pm}} = -iKH_j^{\pm}(K). \tag{3.18}$$

Here the body is assumed implicitly symmetric about the centerline; thus the wave amplitudes on the right and left sides of the body are the same. In this case, the phase of generated wave satisfies $\varepsilon_2^{+} = \varepsilon_2^{-}$ for the case of heave and $\varepsilon_j^{+} = \varepsilon_j^{-} + \pi$ for the case of sway ($j = 1$) and roll ($j = 3$).

In the diffraction problem, by taking account of the boundary conditions satisfied by φ_0 and φ_4 and Green's theorem for φ_0, we can transform the integral equation for the diffraction potential and the resultant calculation formula for the Kochin function in a more convenient form. Getting back to Green's theorem given by (2.43) and considering $\varphi_D = \varphi_0 + \varphi_4$ as the velocity potential, we may obtain the following:

$$\begin{aligned}\varphi_D(\mathrm{P}) = &-\int_{S_H}\varphi_D(\mathrm{Q})\frac{\partial}{\partial n_Q}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q})\\ &+\int_{S_{\pm\infty}}\Big\{\frac{\partial\varphi_0(\mathrm{Q})}{\partial n_Q} - \varphi_0(\mathrm{Q})\,\frac{\partial}{\partial n_Q}\Big\}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}).\end{aligned} \tag{3.19}$$

Here we have taken into account the body boundary condition (3.8) and that the scattering potential φ_4 satisfies the radiation condition of the wave outgoing on both sides of the body but the incident-wave potential φ_0 does not. The second line in (3.19) is the result of φ_0 not satisfying the radiation condition. However this second line is equal to $\varphi_0(\mathrm{P})$ itself on account of Green's theorem for the case of no floating body. Thus we can have the following expression

$$\varphi_4(\mathrm{P}) = -\int_{S_H}\varphi_D(\mathrm{Q})\frac{\partial}{\partial n_Q}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}), \tag{3.20}$$

when the field point P is in the fluid region.

In the same manner, when the field point P is take on the body surface, the coefficient on the left-hand side of (3.19) must be equal to 1/2 and thus the integral equation for φ_D takes the following form:

$$\frac{1}{2}\,\varphi_D(\mathrm{P}) + \int_{S_H}\varphi_D(\mathrm{Q})\frac{\partial}{\partial n_Q}G(\mathrm{P};\mathrm{Q})\,ds(\mathrm{Q}) = \varphi_0(\mathrm{P}). \tag{3.21}$$

Once the velocity potential φ_D has been determined, the asymptotic expression of $\varphi_4(\mathrm{P})$ far from the body can be obtained from (3.20) in the form

$$\left.\begin{aligned}&\varphi_4(\mathrm{P}) \sim iH_4^{\pm}(K)\,e^{-Ky\mp iKx} \quad \text{as } x \to \pm\infty\\ &H_4^{\pm}(K) = -\int_{S_H}\varphi_D\,\frac{\partial}{\partial n}\,e^{-K\eta\pm iK\xi}\,ds(\xi,\eta)\end{aligned}\right\} \tag{3.22}$$

The numerical value from this expression must be identical to that from (3.13) for $j = 4$.

3.3 Calculation of Hydrodynamic Forces

Suppose that the velocity potentials on the body surface φ_j ($j = 1 \sim 4$) are obtained by solving an integral equation (2.48) in the radiation and diffraction problems. Then we shall consider how to calculate hydrodynamic forces acting on the body.

Since the force can be obtained by integrating the pressure on the wetted surface of a body, let us consider the pressure first. In the linear theory, higher-order terms in Bernoulli's pressure equation can be discarded, and thus the total pressure with the reference value taken as the atmospheric pressure can be written as

$$P(x, y, t) = -\rho \frac{\partial \Phi}{\partial t} + \rho g y + O(\Phi^2). \tag{3.23}$$

The second term on the right-hand side represents the hydrostatic pressure. This term has nothing to do with the velocity potential. However, since the body oscillates, variation in this hydrostatic pressure due to body motions must be considered; which can be obtained by substituting the y-component of $\boldsymbol{x} = \overline{\boldsymbol{x}} + \boldsymbol{\alpha}(t)$ given by (3.1). Namely

$$y = \overline{y} + \xi_2(t) + \xi_3(t)\, x = \overline{y} + \mathrm{Re}\big[\, \big(X_2 + X_3\, x \big)\, e^{i\omega t} \,\big]. \tag{3.24}$$

For the first term on the right-hand side of (3.23) representing the hydrodynamic pressure, $\Phi(\boldsymbol{x}, t) = \mathrm{Re}\big[\phi(\boldsymbol{x})\, e^{i\omega t}\big]$ and (3.7) for $\phi(\boldsymbol{x})$ will be substituted. The result may be written in the form

$$\left.\begin{aligned} P(\boldsymbol{x}, t) &= \mathrm{Re}\big[\, p(\boldsymbol{x})\, e^{i\omega t} \,\big] \\ p(\boldsymbol{x}) &= p_R(\boldsymbol{x}) + p_D(\boldsymbol{x}) + p_S(\boldsymbol{x}) \end{aligned}\right\} \tag{3.25}$$

where

$$p_R(\boldsymbol{x}) = -\rho i\omega \sum_{j=1}^{3} i\omega X_j\, \varphi_j(\boldsymbol{x}), \tag{3.26}$$

$$p_D(\boldsymbol{x}) = -\rho i\omega \frac{ga}{i\omega} \Big\{\, \varphi_0(\boldsymbol{x}) + \varphi_4(\boldsymbol{x}) \Big\} = -\rho g a\, \varphi_D(\boldsymbol{x}), \tag{3.27}$$

$$p_S(\boldsymbol{x}) = \rho g \big(\, X_2 + X_3 x \,\big). \tag{3.28}$$

First we shall consider the force to be obtained by integrating the pressure (3.26) in the radiation problem. Noting that the normal vector is defined as positive when directing from the body surface into the fluid, the hydrodynamic force in the i-th direction can be obtained as follows:

$$F_i = -\int_{S_H} p_R(\boldsymbol{x}) n_i\, ds = \rho (i\omega)^2 \sum_{j=1}^{3} X_j \int_{S_H} \varphi_j(\boldsymbol{x}) n_i\, ds \equiv \sum_{j=1}^{3} f_{ij}. \tag{3.29}$$

Here the velocity potential $\varphi_j(\boldsymbol{x})$ is generally given in complex (its imaginary part exists because the Green function in the integral equation includes the imaginary part associated with generation of progressive waves on the free surface). Therefore, by introducing a notation of $\varphi_j(\boldsymbol{x}) = \varphi_{jc}(\boldsymbol{x}) + i\, \varphi_{js}(\boldsymbol{x})$, f_{ij} in (3.29) can be written as

$$\begin{aligned} f_{ij} &= \rho (i\omega)^2 X_j \int_{S_H} \Big\{\, \varphi_{jc}(\boldsymbol{x}) + i\, \varphi_{js}(\boldsymbol{x}) \Big\} n_i\, ds \\ &= -(i\omega)^2 X_j \underbrace{\Big[-\rho \int_{S_H} \varphi_{jc}(\boldsymbol{x}) n_i\, ds \Big]}_{A_{ij}} - i\omega X_j \underbrace{\Big[\, \rho\omega \int_{S_H} \varphi_{js}(\boldsymbol{x}) n_i\, ds \Big]}_{B_{ij}} \end{aligned} \tag{3.30}$$

Thus f_{ij} can be interpreted as the force component acting in the i-th direction due to the j-th mode of motion. Together with time-dependent term $e^{i\omega t}$, we can see that $(i\omega)^2 X_j$ represents the acceleration and $i\omega X_j$ the velocity and hence A_{ij} and B_{ij} can be defined as **the added mass** and **the damping coefficient**, respectively. It should be noted these quantities are defined with minus sign in the acceleration and velocity as in (3.30), because the radiation force is a component of the total force to be considered on the right-hand side of the motion equation in Newton's second law and will be transposed finally onto the left-hand side of the motion equation. The force component f_{ij} may be written as $f_{ij} = T_{ij} X_j$, where T_{ij} can be regarded as the transfer function with respect to the displacement X_j in the j-th mode of motion and given in the form

$$\left.\begin{aligned} F_i = \sum_{j=1}^{3} T_{ij} X_j, \quad T_{ij} = -(i\omega)^2 \Big\{ A_{ij} + \frac{1}{i\omega} B_{ij} \Big\} \\ A_{ij} + \frac{1}{i\omega} B_{ij} = -\rho \int_{S_H} \varphi_j(\boldsymbol{x}) n_i \, ds = -\rho \int_{S_H} \varphi_j(\boldsymbol{x}) \frac{\partial \varphi_i}{\partial n} \, ds \end{aligned}\right\} \tag{3.31}$$

The body boundary condition for φ_i given by (3.9) has been substituted in the last expression.

Next we shall consider the force to be obtained by integrating the pressure (3.27) in the diffraction problem. The resulting hydrodynamic force is referred to as **the wave-exciting force**, and the force acting in the i-th direction can be calculated by

$$\begin{aligned} E_i &= -\int_{S_H} p_D(\boldsymbol{x}) n_i \, ds = \rho g a \int_{S_H} \varphi_D(\boldsymbol{x}) n_i \, ds \\ &= \rho g a \int_{S_H} \Big\{ \varphi_0(\boldsymbol{x}) + \varphi_4(\boldsymbol{x}) \Big\} n_i \, ds. \end{aligned} \tag{3.32}$$

Here the force component related to the incident wave, $\varphi_0(\boldsymbol{x})$, is called **Froude-Krylov force**, which was only the component of wave-exciting force considered in the beginning of the 20th century. With the advent of computers, the effect of wave scattering could be computed and its importance became realized.

Lastly we shall consider the restoring force to be obtained by integrating the variance in the hydrostatic pressure (3.28) due to body motions. In the same way as that for the radiation and diffraction forces, the final formulae may be obtained by the line integral on the wetted surface of a body. However, as an effective alternative, Gauss' theorem will be used here. We should note that the hydrostatic forces act only in the vertical direction in parallel to the gravity in space; that is, contributions exist only in heave and also in roll as the moment due to couple of vertical forces.

The restoring force in heave can be obtained as follows:

$$\begin{aligned} S_2 &= -\int_{S_H} p_S(\boldsymbol{x}) n_2 \, ds = -\rho g \int_{S_H} \big(X_2 + X_3 \, x \big) n_2 \, ds \\ &= -\rho g \int_{-B/2}^{B/2} \big(X_2 + X_3 \, x \big) \, dx = -\rho g B \, X_2 \equiv -C_{22} \, X_2, \end{aligned} \tag{3.33}$$

where B denotes the breadth of a floating body.

Similarly, the restoring moment in roll (about the origin of the coordinate system) becomes

$$\begin{aligned} S_3 &= -\int_{S_H} p_S(\boldsymbol{x}) n_3 \, ds = -\rho g \int_{S_H} \big(X_2 + X_3 \, x \big) \big(n_2 x - n_1 y \big) \, ds \\ &= -\rho g X_3 \int_{-B/2}^{B/2} x^2 \, dx + \rho g X_3 \iint_V y \, dS \\ &= \rho g X_3 \big\{ -\nabla \overline{BM} + \nabla \overline{OB} \big\} = -\rho g \nabla \overline{OM} \, X_3, \end{aligned} \tag{3.34}$$

where the Gauss theorem has been used in transformation, and ∇ denotes the displacement volume, $\overline{BM}$ the vertical length between the center of buoyancy and the metacenter, and $\overline{OB}$ the vertical length between the origin of the coordinate system and the center of buoyancy.

In the motion equation, the moment about the center of gravity (denoted with G) may be needed, which can be computed as follows:

$$
\begin{aligned}
S_3^G &= -\int_{S_H} p_S(\boldsymbol{x}) n_3^G \, ds = -\rho g \int_{S_H} \big(X_2 + X_3\, x \big) \big\{ n_2 x - n_1 (y - \overline{OG}) \big\} \, ds \\
&= S_3 - \rho g \, \overline{OG} \int_{S_H} \big(X_2 + X_3\, x \big) n_1 \, ds = S_3 - \rho g \, \overline{OG} \, X_3 \iint_V dS \\
&= -\rho g \nabla \big\{ \overline{OM} + \overline{OG} \big\} X_3 = -\rho g \nabla \, \overline{GM} \, X_3 \equiv -C_{33} X_3 .
\end{aligned} \tag{3.35}
$$

Here the center of gravity is assumed to be located below the free surface and hence $\overline{GM} = \overline{OG} + \overline{OM}$.

The hydrodynamic force and moment related to the roll motion in the radiation and diffraction problems must also be evaluated about the center of gravity in considering the equations of body motion, which will be described later when we shall consider them.

3.4 Reflection and Transmission Waves

The concept of reflection and transmission waves will be important in considering the deformation of incident wave due to the presence of a body. Details of the characteristics of these waves will be explained subsequently in connection with various hydrodynamic relations between the waves generated by a floating body and hydrodynamic forces acting on that body. In this subsection, only the definition of the reflection and transmission waves and their notations will be described.

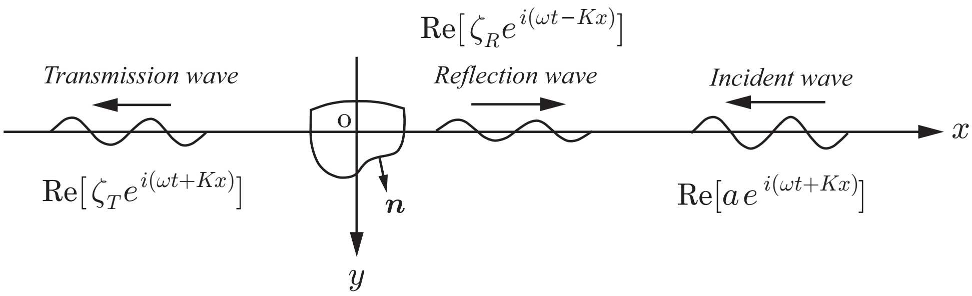


Fig.3.3 Case of incident wave incoming from the positive x-axis

First let us consider the case where the incident wave is incoming from the positive x-axis (see Fig. 3.3). For generality, the shape of a floating body is assumed to be asymmetric in left and right. In this case, the reflection wave can be defined as the wave propagating to the positive x-axis, opposite to the incident wave, and hence from (3.14) the complex amplitude ζ_R can be expressed as

$$
\zeta_R = i a \, H_4^+(K) - iK \sum_{j=1}^{3} X_j^+ \, H_j^+(K). \tag{3.36}
$$

It should be noted that the complex motion amplitude in the case of Fig. 3.3 is written as X_j^+ to distinguish from the case when the direction of incident wave is opposite (which will be considered next).

By dividing with the incident-wave amplitude, the nondimensional reflection wave is defined as

$$
C_R \equiv \frac{\zeta_R}{a} = R - iK \sum_{j=1}^{3} \frac{X_j^+}{a} \, H_j^+(K), \tag{3.37}
$$

where

$$R = iH_4^+(K) \tag{3.38}$$

is the reflection-wave (complex) coefficient when the body is fixed in space (in the diffraction problem), whereas C_R is the corresponding coefficient when the body oscillates in an incident wave.

The transmission wave is defined as the wave passing the body and propagating to the infinity of negative x-axis, which includes the incident wave. Thus the complex amplitude ζ_T and its nondimensional form are expressed in the form

$$\zeta_T = a\Big\{1 + iH_4^-(K)\Big\} - iK\sum_{j=1}^{3} X_j^+ H_j^-(K), \tag{3.39}$$

$$C_T \equiv \frac{\zeta_T}{a} = T - iK\sum_{j=1}^{3}\frac{X_j^+}{a} H_j^-(K), \tag{3.40}$$

where

$$T = 1 + iH_4^-(K) \tag{3.41}$$

is the coefficient of transmission wave when the body is fixed in space.

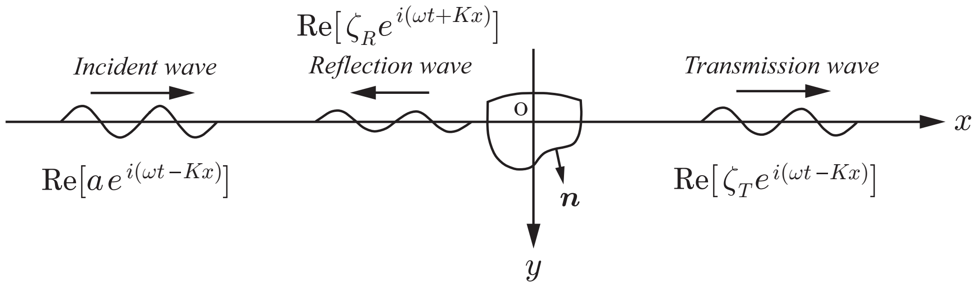


Fig.3.4 Case of incident wave incoming from the negative x-axis

Next let us consider the case where the incident wave is incoming from the negative x-axis (see Fig. 3.4). When the body is asymmetric, the scattered wave (Kochin function in the diffraction problem) is different depending on the direction of the incident wave. To indicate this difference clearly, the Kochin function in the diffraction problem of Fig. 3.4 is denoted by $h_4^{\pm}(K)$ in contrast to $H_4^{\pm}(K)$ for the case of Fig. 3.3.

If the body is asymmetric, the wave-exciting force and hence the complex amplitude of the body motion may also be different depending on the direction of the incident wave. Thus, for the case of Fig. 3.4, the complex amplitude in the j-th mode of motion will be denoted as X_j^-. On the other hand, the Kochin function in the radiation problem is independent of the incident wave. With all these taken into account, the complex coefficient of reflection wave for the case of Fig. 3.4 is given in the form

$$C_R \equiv \frac{\zeta_R}{a} = R - iK\sum_{j=1}^{3}\frac{X_j^-}{a} H_j^-(K), \tag{3.42}$$

where

$$R = ih_4^-(K). \tag{3.43}$$

Likewise, the coefficient of transmission wave for the case of Fig. 3.4 can be expressed as

$$C_T \equiv \frac{\zeta_T}{a} = T - iK\sum_{j=1}^{3}\frac{X_j^-}{a} H_j^+(K), \tag{3.44}$$

where

$$T = 1 + ih_4^+(K). \tag{3.45}$$

Now that the definition of the reflection and transmission waves has been given, we shall consider hydrodynamic relations in connection with these waves by using Green's theorem.

3.5 Hydrodynamic Relations Derived with Green's Theorem

Green's theorem has been applied in subsection 2.2 to derive an expression for the velocity potential in the form of boundary integral. A result from Green's theorem is given by (2.40), which can be written for two different velocity potentials, ϕ and ψ, in the form

$$\oint_S \left\{ \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} \right\} ds = \iint_V \left(\psi\,\nabla^2\phi - \phi\,\nabla^2\psi \right) dS. \tag{3.46}$$

Here, as shown in Fig. 3.5, the boundary S surrounding the fluid region V consists of the hull surface of a floating body S_H, the free surface S_F, the radiation surface far from the body $S_{\pm\infty}$, and the bottom of water S_B.

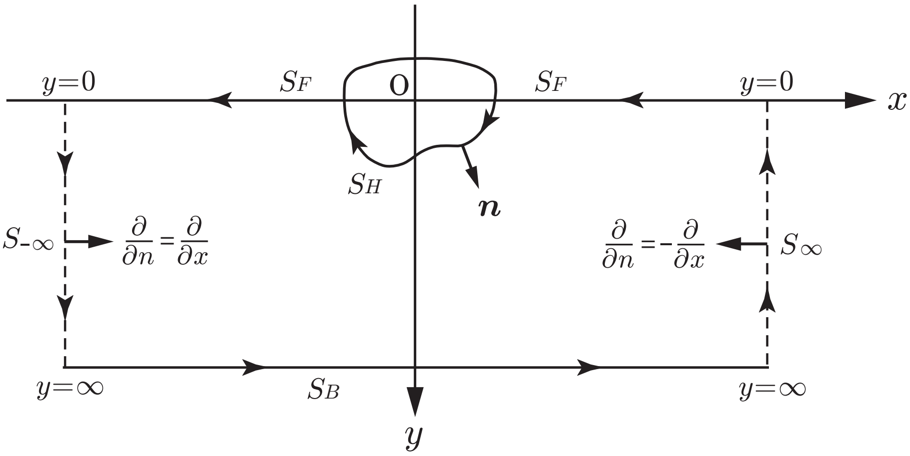


Fig.3.5 Application of Green's theorem

We note that the velocity potentials ϕ and ψ in (3.46) are regular in the fluid region and hence satisfy $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$. Namely the right-hand side of (3.46) is zero. Furthermore we assume in what follows that both ϕ and ψ satisfy the same boundary conditions on S_F and S_B but not necessarily the same on $S_{\pm\infty}$ and S_H. In this case, the line integrals on S_F and S_B on the left-hand side of (3.46) become zero and thus

$$\int_{S_H+S_{\pm\infty}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} \right) ds = 0. \tag{3.47}$$

Since the radiation boundaries are parallel to the y-axis, the following relations hold

$$\left.\begin{aligned} &\text{on } S_{-\infty}, \quad \frac{\partial}{\partial n} = \frac{\partial}{\partial x}, \quad ds = dy, \quad y: 0 \to \infty \\ &\text{on } S_{+\infty}, \quad \frac{\partial}{\partial n} = -\frac{\partial}{\partial x}, \quad ds = -\,dy, \quad y: \infty \to 0 \end{aligned}\right\} \tag{3.48}$$

Therefore (3.47) can be written as follows:

$$\begin{aligned} \int_{S_H} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} \right) ds &= -\left[\int_{S_{+\infty}} + \int_{S_{-\infty}} \right] \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} \right) ds \\ &= \int_0^\infty dy \left[\phi\frac{\partial\psi}{\partial x} - \psi\frac{\partial\phi}{\partial x} \right]_{x=-\infty}^{x=+\infty} \end{aligned} \tag{3.49}$$

where [   ] in the last expression of (3.49) denotes the difference of the quantity in brackets between the values at $x = +\infty$ and at $x = -\infty$.

The y-dependence in the velocity potentials far from the body is, as explicitly shown in (3.12), expressed as e^{-Ky}. Thus the integral with respect to y appearing in (3.49) may be performed in advance, and the result takes the form

$$\int_0^\infty e^{-2Ky}\,dy = \frac{1}{2K}. \tag{3.50}$$

With this result, (3.49) can be written as

$$\int_{S_H}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = \frac{1}{2K}\left[\left(\phi\frac{\partial\psi}{\partial x} - \psi\frac{\partial\phi}{\partial x}\right)_{y=0}\right]_{x=-\infty}^{x=+\infty} \tag{3.51}$$

This is the base equation for deriving various hydrodynamic relations in subsequent sections.

3.5.1 Symmetry relations in the added mass and damping coefficient

We may choose any combination for ϕ and ψ, as long as velocity potentials satisfy the same boundary conditions on S_F and S_B. As the first choice, let us set $\phi = \varphi_i$ and $\psi = \varphi_j$ in the radiation problem. Since both satisfy the same radiation condition of outgoing waves, the right-hand side of (3.51) must be zero, which may be confirmed directly by substituting asymptotic forms of φ_i and φ_j:

$$\left.\begin{aligned}\varphi_i(x,0) &\sim i\,H_i^{\pm}\,e^{\mp iKx}\\ \varphi_j(x,0) &\sim i\,H_j^{\pm}\,e^{\mp iKx}\end{aligned}\right\}\quad \text{as } x\to\pm\infty \tag{3.52}$$

Therefore it follows that

$$\int_{S_H}\varphi_i\frac{\partial\varphi_j}{\partial n}\,ds = \int_{S_H}\varphi_j\frac{\partial\varphi_i}{\partial n}\,ds, \tag{3.53}$$

and from (3.31) this relation is equivalent to writing as

$$T_{ji} = T_{ij}. \tag{3.54}$$

Separating this expression into the real and imaginary parts, we have

$$A_{ji} = A_{ij}, \qquad B_{ji} = B_{ij}, \tag{3.55}$$

which represents the symmetry relations in the added-mass and damping coefficients; that is, the radiation force acting in the j-th direction due to the i-th mode of motion is equal to the corresponding value acting in the i-th direction due to the j-th mode of motion.

3.5.2 Relations of energy conservation

Next let us take $\phi = \varphi_i$ and $\psi = \overline{\varphi_j}$ (where the overbar means the complex conjugate) and consider the consequence of (3.51). We note that the complex conjugate of the velocity potential can be interpreted as the **reverse-time velocity potential** (which was named first by Bessho), because this velocity potential with time-dependent part $e^{i\omega t}$ can also be written in the form

$$\varPhi(\boldsymbol{x},t) = \mathrm{Re}\big[\,\overline{\varphi_j}(\boldsymbol{x})\,e^{i\omega t}\,\big] = \mathrm{Re}\big[\,\varphi_j(\boldsymbol{x})\,e^{-i\omega t}\,\big]. \tag{3.56}$$

Thus considering the complex conjugate of the spatial part is equivalent to considering the original spatial part with time reversed. We also note that the body boundary condition in the radiation problem is given by (3.9) and the normal vector is of real quantity. Thus we have the followings for φ_i and $\overline{\varphi_j}$

$$\frac{\partial\varphi_i}{\partial n} = n_i, \quad \frac{\partial\overline{\varphi_j}}{\partial n} = \overline{n_j} = n_j \qquad \text{on } S_H. \tag{3.57}$$

Since the transfer function defined by (3.31) is of complex quantity, the left-hand side of (3.51) can be written as

$$\begin{aligned}\mathcal{L} \equiv \int_{S_H} \left(\varphi_i \frac{\partial \overline{\varphi_j}}{\partial n} - \overline{\varphi_j} \frac{\partial \varphi_i}{\partial n} \right) ds &= \int_{S_H} \varphi_i \, n_j \, ds - \int_{S_H} \overline{\varphi_j \, n_i} \, ds \\ &= -\frac{1}{\rho} \left\{ A_{ji} + \frac{1}{i\omega} B_{ji} - \left(A_{ij} - \frac{1}{i\omega} B_{ij} \right) \right\} = \frac{2i}{\rho\omega} B_{ij}, \end{aligned} \tag{3.58}$$

where the symmetry relations (3.55) have been taken into account in the last expression of (3.58).

On the other hand, the right-hand side of (3.51) can be evaluated with the following asymptotic expressions:

$$\left.\begin{aligned} \varphi_i(x,0) \sim i\, H_i^{\pm} \, e^{\mp iKx}, \qquad & \frac{\partial \varphi_i}{\partial x} \sim \pm K \, H_i^{\pm} \, e^{\mp iKx} \\ \overline{\varphi_j}(x,0) \sim -i\, \overline{H_j^{\pm}} \, e^{\pm iKx}, \quad & \frac{\partial \overline{\varphi_j}}{\partial x} \sim \pm K \, \overline{H_j^{\pm}} \, e^{\pm iKx} \end{aligned}\right\} \tag{3.59}$$

The result after substituting these takes the following form:

$$\mathcal{R} \equiv \frac{1}{2K} \left[\left(\varphi_i \frac{\partial \overline{\varphi_j}}{\partial x} - \overline{\varphi_j} \frac{\partial \varphi_i}{\partial x} \right)_{y=0} \right]_{x=-\infty}^{x=+\infty} = i \left\{ H_i^{+} \overline{H_j^{+}} + H_i^{-} \overline{H_j^{-}} \right\}. \tag{3.60}$$

Therefore $\mathcal{L} = \mathcal{R}$ gives the following relation:

$$B_{ij} = \frac{1}{2} \rho\omega \left\{ H_i^{+} \overline{H_j^{+}} + H_i^{-} \overline{H_j^{-}} \right\}. \tag{3.61}$$

For a body with port and starboard symmetry, the Kochin function in the radiation problem satisfies a relation of $H_j^{-} = (-1)^j H_j^{+}$ and hence

$$B_{22} = \rho\omega \left| \, H_2^{+} \, \right|^2, \tag{3.62}$$

$$B_{ij} = \rho\omega \, H_i^{+} \, \overline{H_j^{+}} \quad \text{for } i, j = 1 \text{ or } 3. \tag{3.63}$$

These are known as the energy conservation, relating the damping coefficient to the square of the amplitude of body-generated progressive wave.

This relation of energy conservation can be derived in another way. The work done by the body motion in the j-th mode on the fluid can be obtained by taking time average of the pressure integral as follows:

$$\begin{aligned} W_D = \int_{S_H} \overline{P \, V_n} \, ds &= \int_{S_H} \overline{\mathrm{Re}\big[\, p_R(\boldsymbol{x}) \, e^{i\omega t} \, \big] \, \mathrm{Re}\big[\, i\omega X_j \, n_j \, e^{i\omega t} \, \big]} \, ds \\ &= \frac{1}{2} \, \mathrm{Re} \int_{S_H} p_R(\boldsymbol{x}) \big(- i\omega X_j \, n_j \, \big) \, ds. \end{aligned} \tag{3.64}$$

Here V_n denotes the normal velocity of the body, and a formula for taking time average of the product of two different quantities in harmonic oscillation has been applied. Since the pressure in the radiation problem is given by (3.26) with $\varphi_j(\boldsymbol{x}) = \varphi_{jc}(\boldsymbol{x}) + i\varphi_{js}(\boldsymbol{x})$, the result of (3.64) can be expressed as

$$\begin{aligned} W_D &= \frac{1}{2} \, \mathrm{Re} \left[\rho \, (i\omega)(i\omega X_j)^2 \int_{S_H} \big\{ \varphi_{jc}(\boldsymbol{x}) + i\varphi_{js}(\boldsymbol{x}) \big\} n_j \, ds \right] \\ &= \frac{1}{2} \, (\omega X_j)^2 \rho\omega \int_{S_H} \varphi_{js}(\boldsymbol{x}) \, n_j \, ds = \frac{1}{2} \, (\omega X_j)^2 B_{jj}, \end{aligned} \tag{3.65}$$

where the definition of the damping coefficient in (3.30) has been used. We can see from this result that only the damping force contributes to the work and the inertia force does not.

This work is imparted to the fluid and must be equal to the mean rate of energy flux of progressive waves generated by the body. The energy density of progressive wave with amplitude a is $\frac{1}{2}\rho g a^2$ and the amplitude of the radiation wave is given by (3.16). Therefore the energy density of outgoing waves on both sides of the body is given by

$$\left.\begin{aligned} E^{+} &= \frac{1}{2}\rho g \big| K X_j H_j^{+} \big|^2 = \frac{1}{2}\rho\omega(\omega X_j)^2 \big| H_j^{+} \big|^2 \frac{\omega}{g} \\ E^{-} &= \frac{1}{2}\rho g \big| K X_j H_j^{-} \big|^2 = \frac{1}{2}\rho\omega(\omega X_j)^2 \big| H_j^{-} \big|^2 \frac{\omega}{g} \end{aligned}\right\} \tag{3.66}$$

The velocity of the energy flux is equal to the group velocity (which is given by $c_g = \frac{1}{2}\frac{\omega}{K} = \frac{1}{2}\frac{g}{\omega}$ in deep water), and the rate of change of the total energy is the product of $(E^{+} + E^{-})$ and c_g. Thus we have

$$\overline{\frac{dE}{dt}} = \big(E^{+} + E^{-} \big) c_g = \frac{1}{2}(\omega X_j)^2 \frac{1}{2}\rho\omega \Big\{ \big| H_j^{+} \big|^2 + \big| H_j^{-} \big|^2 \Big\}. \tag{3.67}$$

Equating (3.65) and (3.67), we can obtain the relation (3.61) for the case of $i = j$.

The energy conservation in the diffraction problem may be derived in the same way. Let us consider a combination of $\phi = \varphi_D$ and $\psi = \overline{\varphi_D}$. In this case, with the body boundary condition (3.8), the integral on the body surface becomes zero, and thus only the right-hand side of (3.51) must be zero. That is,

$$\left[\left(\varphi_D \frac{\partial \overline{\varphi_D}}{\partial x} - \overline{\varphi_D} \frac{\partial \varphi_D}{\partial x} \right)_{y=0} \right]_{x=-\infty}^{x=+\infty} = 0. \tag{3.68}$$

Considering the case of Fig. 3.3 (incident wave incoming from the positive x-axis), the asymptotic form of φ_D can be written in terms of the reflection and transmission wave coefficients in the form

$$\left.\begin{aligned} &\varphi_D(x,0) \sim e^{iKx} + R\, e^{-iKx} && \text{as } x \to +\infty \\ &\varphi_D(x,0) \sim T\, e^{iKx} && \text{as } x \to -\infty \end{aligned}\right\} \tag{3.69}$$

Substituting these into (3.68) gives the following result:

$$\begin{aligned} -iK\big(e^{iKx} + R\, e^{-iKx} \big)\big(e^{-iKx} - \overline{R}\, e^{iKx} \big) - iK\big(e^{-iKx} + \overline{R}\, e^{iKx} \big)\big(e^{iKx} - R\, e^{-iKx} \big) \\ - \big(-iK\, T\overline{T} - iK\, T\overline{T} \big) = 0. \end{aligned}$$

Therefore

$$\big| R \big|^2 + \big| T \big|^2 = 1. \tag{3.70}$$

Since R and T are the coefficients nondimensionalized with the incident-wave amplitude a, (3.70) can be written in the form

$$\frac{1}{2}\rho g \big| \zeta_R \big|^2 + \frac{1}{2}\rho g \big| \zeta_T \big|^2 = \frac{1}{2}\rho g a^2. \tag{3.71}$$

The right-hand side, $\frac{1}{2}\rho g a^2$, represents the energy density of incident wave as the input. Thus (3.71) tells us that the total energy density after diffraction by the body remains the same as that of the input, which means the energy conservation in the diffraction problem.

3.5.3 Haskind-Newman's relation

Hydrodynamic relations between the radiation and diffraction problems may be derived by considering a combination of $\phi = \varphi_D$ and $\psi = \varphi_j$ $(j = 1, 2, 3)$. Taking account of the body boundary conditions (3.8) for φ_D and (3.9) for φ_j and the calculation formula for the wave-exciting force (3.32), the left-hand side of (3.51) gives the following result:

$$\mathcal{L} = \int_{S_H} \left(\varphi_D \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_D}{\partial n} \right) ds = \int_{S_H} \varphi_D\, n_j\, ds = \frac{E_j}{\rho g a}. \tag{3.72}$$

On the other hand, the right-hand side of (3.51) can be evaluated with the following expressions for the velocity potentials:

$$\left.\begin{array}{lll} \text{as } x\to+\infty, & \varphi_D \sim e^{iKx}+R\,e^{-iKx}, & \varphi_j \sim i\,H_j^+\,e^{-iKx} \\ \text{as } x\to-\infty, & \varphi_D \sim T\,e^{iKx}, & \varphi_j \sim i\,H_j^-\,e^{iKx} \end{array}\right\} \tag{3.73}$$

The result of the calculation for the right-hand side takes the form

$$\begin{aligned} \mathcal{R} = \frac{1}{2K}\Big[\,&KH_j^+\big\{\,1+R\,e^{-i2Kx}\,\big\}+KH_j^+\big\{\,1-R\,e^{-i2Kx}\,\big\} \\ &+KH_j^-T\,e^{i2Kx}-KH_j^-T\,e^{i2Kx}\,\Big] = H_j^+. \end{aligned} \tag{3.74}$$

Equating (3.72) and (3.74) gives the following relation

$$E_j = \rho g a\, H_j^+, \tag{3.75}$$

which is known as **Haskind-Newman's relation**.

We note that E_j is the wave-exciting force acting in the j-th direction exerted by the incident wave incoming from the positive x-axis, and H_j^+ is associated with the complex amplitude of the radiation wave generated by the j-th mode of motion, propagating to the positive x-axis (opposite to the direction of propagation of the incident wave). Although the diffraction and radiation problems look superficially unrelated, (3.75) states a remarkable relation that the wave-exciting force E_j can be computed only from the complex wave amplitude H_j^+ in the radiation problem.

Since the damping coefficient can be computed from the square of the Kochin function, (3.61) and (3.75) gives also the following relation:

$$B_{jj} = \rho\omega\left|\frac{E_j}{\rho g a}\right|^2 = \frac{\omega}{\rho g^2}\left|\frac{E_j}{a}\right|^2 = \frac{1}{2\rho g c_g}\left|\frac{E_j}{a}\right|^2. \tag{3.76}$$

Namely the damping coefficient and the wave-exciting force are directly related.

As an easy extension, let us consider next a combination of $\phi=\varphi_D$ and $\psi=\overline{\varphi_j}$. In this case, with (3.57), the left-hand side of (3.51) turns out the same as (3.72). However, the right-hand side of (3.51) will be different from (3.74) and evaluated in terms of (3.73). The result takes the form

$$\begin{aligned} \mathcal{R} &= \frac{1}{2K}\Big[\,K\overline{H}_j^+\big\{\,e^{i2Kx}+R\,\big\}-K\overline{H}_j^+\big\{\,e^{i2Kx}-R\,\big\}+K\overline{H}_j^-T+K\overline{H}_j^-T\,\Big] \\ &= \overline{H}_j^+\,R+\overline{H}_j^-\,T. \end{aligned} \tag{3.77}$$

Therefore the following relation can be obtained:

$$E_j = \rho g a\Big\{\,\overline{H}_j^+\,R+\overline{H}_j^-\,T\,\Big\}. \tag{3.78}$$

This result must be equal to (3.75) and hence we have

$$H_j^+ = \overline{H}_j^+\,R+\overline{H}_j^-\,T. \tag{3.79}$$

We note that $R = i\,H_4^+$ and $T = 1+i\,H_4^-$. Thus (3.79) implies that a relation exists between the scattered wave in the diffraction problem and the radiated wave by forced oscillation in the radiation problem. This relation will be investigated more in the next subsection for a floating body with port and starboard symmetry.

3.5.4 Relation between radiation and diffraction waves

The Kochin function for the radiation wave generated by a symmetric body has the property of $H_j^- = (-1)^j H_j^+$. Thus (3.79) can be written separately for the case of symmetric motion ($j = 2$ for heave) and for the case of antisymmetric motion ($j = 1$ for sway or $j = 3$ for roll) as follows:

$$H_2^+ = \overline{H}_2^+ \big(i\, H_4^+ + 1 + i\, H_4^- \big), \tag{3.80}$$

$$H_j^+ = \overline{H}_j^+ \big(i\, H_4^+ - 1 - i\, H_4^- \big) \quad (j = 1 \text{ or } 3). \tag{3.81}$$

Here we note that the Kochin function for the scattered wave, $H_4^{\pm}(K)$, can be separated into the symmetric and antisymmetric components with respect to the y-axis. More specifically from (3.22), it can be written as

$$\begin{aligned} H_4^{\pm}(K) &= -\int_{S_H} \varphi_D \frac{\partial}{\partial n}\, e^{-K\eta} \cos K\xi \, ds \pm \Big\{ -i \int_{S_H} \varphi_D \frac{\partial}{\partial n}\, e^{-K\eta} \sin K\xi \, ds \Big\} \\ &\equiv C_4(K) \pm S_4(K). \end{aligned} \tag{3.82}$$

First, by substituting (3.82) in (3.80), we have

$$2i\, C_4 + 1 = H_2^+ / \overline{H}_2^+ . \tag{3.83}$$

It can be seen from this that the symmetric component of the scattered wave C_4 is given by the radiation wave generated by the heave motion. In terms of (3.18) as an expression of H_2^+, C_4 can be expressed as follows:

$$C_4 = \mathrm{Im}\big(H_2^+ \big) / \overline{H}_2^+ = i\, e^{i\varepsilon_2} \cos\varepsilon_2 . \tag{3.84}$$

Likewise, substituting (3.82) in (3.81), we have an expression for the antisymmetric component as follows:

$$2i\, S_4 - 1 = H_j^+ / \overline{H}_j^+ \quad (j = 1 \text{ or } 3). \tag{3.85}$$

Thus

$$S_4 = -i\, \mathrm{Re}\big(H_j^+ \big) / \overline{H}_j^+ = -\, e^{i\varepsilon_j} \sin\varepsilon_j . \tag{3.86}$$

By combining (3.84) and (3.86), the Kochin function in the diffraction problem $H_4^{\pm} = C_4 \pm S_4$ can be expressed as

$$H_4^{\pm} = \frac{\mathrm{Im}\big(H_2^+ \big)}{\overline{H}_2^+} \mp i\, \frac{\mathrm{Re}\big(H_j^+ \big)}{\overline{H}_j^+} = i\, e^{i\varepsilon_2} \cos\varepsilon_2 \mp e^{i\varepsilon_j} \sin\varepsilon_j . \tag{3.87}$$

Thus we can see that the scattered wave can be obtained in terms of only the phase of the radiation wave. This relation was proven first by Bessho and later by Newman with different analysis, and thus we call (3.87) **Bessho-Newman's relation**.

3.5.5 Bessho's relation for damping coefficients

In the relations shown above, such as (3.81) and (3.87), the mode index j can be 1 or 3, which suggests that some important relation exists between the Kochin functions for sway and roll.

Considering $j = 1$ and $j = 3$ for (3.81), we can obtain the relation

$$H_1^+ / \overline{H}_1^+ = H_3^+ / \overline{H}_3^+ .$$

Thus

$$H_3^+ / H_1^+ = \overline{H}_3^+ / \overline{H}_1^+ \equiv \ell_w . \tag{3.88}$$

Since a complex quantity equals its conjugate, it must be of real quantity, and the ratio H_3^+ / H_1^+ has a dimension of length; this length (moment lever) of real quantity is denoted as ℓ_w in (3.88). We can see from this relation that the phase of the Kochin functions in sway and roll is exactly the same and that is why the mode index j can be 1 or 3 in (3.87).

As shown by (3.63), the damping coefficient can be computed with the Kochin function. Substituting (3.88) in (3.63), we can find the following relations:

$$B_{13} = B_{31} = B_{11}\,\ell_w, \quad B_{33} = B_{11}\,\ell_w^2. \tag{3.89}$$

We can see that the damping coefficient in roll (B_{33}) can be computed from the damping coefficient in sway (B_{11}) and ℓ_w necessary in this computation can be provided by $\ell_w = B_{31}/\,B_{11}$ only with solutions of the boundary-value problem for sway. The relation (3.89) is also known as **Bessho's relation**.

3.5.6 Reflection and transmission waves by an asymmetric body

In this subsection we consider an asymmetric body; for which, as described in section 3.4, the scattered wave must be different depending on the direction of the incident wave. For the case of Fig. 3.3 (where the incident wave is incoming from the positive x-axis), the Kochin function of the scattered wave was denoted by $H_4^{\pm}(K)$, and for the case of Fig. 3.4 (where the incident wave is incoming from the negative x-axis), the corresponding Kochin function was denoted by $h_4^{\pm}(K)$. In connection with these two cases, the diffraction potential for the case of Fig. 3.3 is denoted as φ_D and the one for the case of Fig. 3.4 will be denoted as ψ_D. Then we consider a combination of $\phi = \varphi_D$ and $\psi = \psi_D$ in Green's theorem (3.51).

Since both cases are the diffraction problem, the left-hand side of (3.51) is equal to zero owing to homogeneous boundary conditions on the body surface. Thus we have

$$\left[\left(\varphi_D \frac{\partial \psi_D}{\partial x} - \psi_D \frac{\partial \varphi_D}{\partial x}\right)_{y=0}\right]_{x=-\infty}^{x=+\infty} = 0. \tag{3.90}$$

Here the asymptotic forms of φ_D and ψ_D at $x = \pm\infty$ are written in terms of the Kochin function in the form

$$\left.\begin{array}{l} \varphi_D(x,0) \sim e^{iKx} + i\,H_4^+\,e^{-iKx} \\ \psi_D(x,0) \sim e^{-iKx} + i\,h_4^+\,e^{-iKx} \end{array}\right\} \quad \text{as } x \to +\infty, \tag{3.91}$$

$$\left.\begin{array}{l} \varphi_D(x,0) \sim e^{iKx} + i\,H_4^-\,e^{iKx} \\ \psi_D(x,0) \sim e^{-iKx} + i\,h_4^-\,e^{iKx} \end{array}\right\} \quad \text{as } x \to -\infty. \tag{3.92}$$

Substitution of these in (3.90) can be written as

$$\begin{aligned} &-iK(\,e^{iKx} + iH_4^+\,e^{-iKx}\,)\,e^{-iKx}(\,1 + ih_4^+\,) - iK(\,e^{iKx} - iH_4^+\,e^{-iKx}\,)\,e^{-iKx}(\,1 + ih_4^+\,) \\ &+iK\,e^{iKx}(\,1 + iH_4^-\,)(\,e^{-iKx} - ih_4^-\,e^{iKx}\,) + iK\,e^{iKx}(\,1 + iH_4^-\,)(\,e^{-iKx} + ih_4^-\,e^{iKx}\,) = 0. \end{aligned}$$

Therefore the result is expressed in a compact form as follows:

$$h_4^+(K) = H_4^-(K) \tag{3.93}$$

As shown by (3.41) and (3.45), H_4^- is associated with the transmission wave in Fig. 3.3 and h_4^+ is associated with the transmission wave in Fig. 3.4. Thus (3.93) means that the complex amplitude of transmission wave (both amplitude and phase) past an asymmetric body must be the same irrespective of the incoming direction of the incident wave.

Next let us consider a combination of $\phi = \overline{\varphi_D}$ and $\psi = \psi_D$. The left-hand side of (3.51) is zero in this case too; thus (3.90) holds with φ_D replaced with $\overline{\varphi_D}$. Using (3.91) and (3.92), the result can be written as

$$
\begin{aligned}
&-iK(\,e^{-iKx} - i\overline{H}_4^{+}\,e^{iKx}\,)\,e^{-iKx}(\,1 + ih_4^{+}\,) + iK(\,e^{-iKx} + i\overline{H}_4^{+}\,e^{iKx}\,)\,e^{-iKx}(\,1 + ih_4^{+}\,)\\
&+iK\,e^{-iKx}(\,1 - i\overline{H}_4^{-}\,)(\,e^{-iKx} - ih_4^{-}\,e^{iKx}\,) - iK\,e^{-iKx}(\,1 - i\overline{H}_4^{-}\,)(\,e^{-iKx} + ih_4^{-}\,e^{iKx}\,) = 0,
\end{aligned}
$$

from which we can obtain the following relation:

$$\overline{H}_4^{+}\big(\,1 + ih_4^{+}\,\big) = h_4^{-}\big(\,1 - i\overline{H}_4^{-}\,\big). \tag{3.94}$$

Taking account of (3.93), this relation can be expressed as

$$h_4^{-}(K) = \overline{H}_4^{+}(K)\,\frac{1 + iH_4^{-}(K)}{1 - i\overline{H}_4^{-}(K)}. \tag{3.95}$$

It is obvious from (3.95) that $\big|\,h_4^{-}(K)\,\big| = \big|\,H_4^{+}(K)\,\big|$. Thus from the definition of the reflection wave (3.38) for the case of Fig. 3.3 and (3.43) for the case of Fig. 3.4, we can see that the amplitude of the reflection wave by an asymmetric body must be the same irrespective of the incoming direction of the incident wave. However, we should note that the phase is different in general depending on the direction of the incident wave.

3.5.7 Energy equally-splitting law

Let us consider a consequence of (3.95) for the case of symmetric bodies. Since $H_4^{\pm} = h_4^{\mp}$ holds for a symmetric body, (3.95) can be expressed as

$$H_4^{+}\big(\,1 - i\overline{H}_4^{-}\,\big) = \overline{H}_4^{+}\big(\,1 + iH_4^{-}\,\big). \tag{3.96}$$

In terms of the coefficients of the reflection wave R defined in (3.38) and the transmission wave T defined in (3.41), (3.96) can be rewritten as

$$R\overline{T} + \overline{R}T = 0. \tag{3.97}$$

Thus we have

$$\mathrm{Re}\big\{\,R\overline{T}\,\big\} = 0. \tag{3.98}$$

On the other hand, as shown by (3.70), $|\,R\,|^2 + |\,T\,|^2 = 1$ was proven as the energy conservation in the diffraction problem. Combining these relations, we can obtain the following relation:

$$\big|\,R \pm T\,\big| = 1. \tag{3.99}$$

This relation may be interpreted as the energy equally-splitting law by the following explanation.

The waves at $x \to +\infty$ and $x \to -\infty$ can be written as $\zeta(x) = R$ and $\zeta(-x) = T$, respectively. Thus the wave can generally be decomposed in the form

$$
\begin{aligned}
\zeta(x) &= \frac{1}{2}\Big\{\,\zeta(x) + \zeta(-x)\,\Big\} + \frac{1}{2}\Big\{\,\zeta(x) - \zeta(-x)\,\Big\}\\
&= \frac{1}{2}\big(\,R + T\,\big) + \frac{1}{2}\big(\,R - T\,\big).
\end{aligned} \tag{3.100}
$$

Namely the first term on the right-hand side, $\frac{1}{2}(\,R+T\,)$, represents the symmetric component of the wave about the body and likewise the second term on the right-hand side, $\frac{1}{2}(\,R-T\,)$, the antisymmetric component of the wave. Therefore, (3.99) implies that the amplitudes of the symmetric and anti-symmetric wave components about the body are the same. Since the wave amplitude is connected with the energy density of incident wave, (3.99) tells us that the energy density of incident wave as the input will be split equally into the energy density of symmetric and antisymmetric waves after the diffraction by the body. The proof shown here is only for the case of fixed body, but the same is true even for the case of a symmetric body freely oscillating in waves; which will be described in the next chapter.

3.6 Characteristics of Wave-Induced Motions of a Floating Body

Various relations on hydrodynamic forces and body-generated waves have been derived. By taking account of those relations, let us study the characteristics of wave-induced motions of a floating body. To give the result analytically in a compact form and thereby understand the essence of the theory, a floating body is assumed symmetric with respect to the centerline. For this case, the symmetric (heave) and antisymmetric (sway and roll) modes of motion are not coupled, and thus they can be treated independently.

First, let us consider heave as the symmetric mode of motion. The equation of motion can be provided by the Newton's second law, because all forces on a body have been analyzed using the inertial coordinate system. The external forces to be considered as the right-hand side of the motion equation are the radiation force, the wave-exciting force and the restoring force due to variation in the hydrostatic pressure. Considering only the harmonic motion with circular frequency ω and denoting the heave motion as $\xi_2(t) = \mathrm{Re}\{\, X_2\, e^{i\omega t}\}$, the motion equation for the complex amplitude can be written as follows:

$$\begin{aligned} m\,(i\omega)^2 X_2 &= F_2 + E_2 + S_2 \\ &= -\{\,(i\omega)^2 A_{22} + i\omega B_{22}\,\}X_2 + E_2 - C_{22}X_2. \end{aligned}$$

Therefore

$$\Big[\, C_{22} - \omega^2\big(\, m + A_{22}\,\big) + i\omega B_{22}\,\Big] X_2 = E_2. \tag{3.101}$$

Here m denotes the mass of floating body. For the coefficients appearing in (3.101), the following relations have been obtained:

(3.62)	$B_{22} = \rho\omega \big\| H_2^+(K) \big\|^2$	Relation of energy conservation
(3.75)	$E_2 = \rho g a H_2^+(K)$	Haskind-Newman's relation
(3.18)	$H_2^+(K) = \dfrac{i}{K}\bar{A}_2\, e^{i\varepsilon_2}$	Expression of Kochin function

Substituting these and adopting the following notations

$$\left.\begin{aligned} & C_{22} - \omega^2(\, m + A_{22}\,) \equiv \rho\omega^2 E^2, \\ & \mathrm{Tan}^{-1}\frac{E^2}{|\, H_2^+\,|^2} \equiv \alpha_H\,, \quad H_{2E}^+ = \frac{H_2^+}{E}\,, \end{aligned}\right\} \tag{3.102}$$

the solution for the complex amplitude can be written in the form

$$\frac{X_2}{a} = \frac{1}{KE}\frac{H_{2E}^+}{\{\, 1 + i|H_{2E}^+|^2\}} = \frac{|\cos\alpha_H\,|}{iK\overline{H}_2^+}\, e^{i\alpha_H} \tag{3.103}$$

$$= \frac{|\cos\alpha_H\,|}{\bar{A}_2}\, e^{i\delta_2}, \quad \delta_2 = \alpha_H + \varepsilon_2. \tag{3.104}$$

Thus we can see that the amplitude is inversely proportional to the wave amplitude ratio $\bar{A}_2$ and the phase of motion relative to the incident wave is given by $\delta_2 = \alpha_H + \varepsilon_2$. At resonance, the restoring and inertial forces are balanced and thus from (3.102) $E = 0$ and $\alpha_H = 0$. In this case, from (3.104) $X_2/a = 1/\bar{A}_2$ and $\delta_2 = \varepsilon_2$. We can see also in the limit of $\omega \to 0$ (i.e. long wavelength) that $H_2^+ \to B$ and $E^2 \to B/K$ (because $E_2 \to \rho g a B$ and $C_{22} = \rho g B$) and thus $\bar{A}_2 \to KB$, $\varepsilon_2 \to -\pi/2$, and $\alpha_H \to \pi/2$; then $X_2/a \to 1$ and $\delta_2 \to 0$.

Next, as the antisymmetric motion, let us consider the coupled motions of sway and roll. In the same way as that for heave, the equations of coupled motions are expressed as

$$\left.\begin{aligned} m\,(i\omega)^2 X_1 &= F_1 + E_1\,,\\ I_R\,(i\omega)^2 X_3 &= F_3 + E_3 + S_3\,, \end{aligned}\right\} \tag{3.105}$$

where I_R denotes the moment of inertia in roll, and in terms of various relations already proven, the force components on the right-hand side can be written as follows:

$$\left.\begin{array}{c} F_1 = T_{11}\,X_1 + T_{13}\,X_3\,,\ F_3 = T_{31}\,X_1 + T_{33}\,X_3\,,\ T_{ij} = -(i\omega)^2 A_{ij} - i\omega B_{ij}\,,\\ A_{13} = A_{31}\,,\ B_{13} = B_{31} = B_{11}\ell_w, B_{33} = B_{11}\ell_w^2\,,\ B_{11} = \rho\omega|H_1^+|^2\,,\\ E_1 = \rho g a H_1^+\,,\ E_3 = \rho g a H_3^+ = \rho g a H_1^+\ell_w\,,\ S_3 = -C_{33}X_3\,. \end{array}\right\} \tag{3.106}$$

Substituting these in (3.105) and rearranging, the result may be written in the form

$$\Big[\,S^2 + i\,|H_1^+|^2\,\Big]X_1 + \Big[\,Q^2 + i\,|H_1^+|^2\,\Big]\ell_w X_3 = \frac{a}{K}\,H_1^+\,, \tag{3.107}$$

$$\Big[\,Q^2 + i\,|H_1^+|^2\,\Big]\ell_w X_1 + \Big[\,R^2 + i\,|H_1^+|^2\,\Big]\ell_w^2 X_3 = \frac{a}{K}\,H_1^+\ell_w\,, \tag{3.108}$$

where S, Q, and R are defined as

$$\left.\begin{aligned} &-\omega^2(\,m + A_{11}\,) \equiv \rho\omega^2 S^2\,,\\ &-\omega^2 A_{13} = -\omega^2 A_{31} \equiv \rho\omega^2 Q^2\ell_w\,,\\ &C_{33} - \omega^2(\,I_R + A_{33}) \equiv \rho\omega^2 R^2\ell_w^2\,. \end{aligned}\right\} \tag{3.109}$$

To write the solution in a compact form corresponding to (3.103) and (3.104), let us define the following symbols:

$$\left.\begin{aligned} F^2 &\equiv \frac{S^2R^2 - Q^4}{S^2 + R^2 - 2Q^2}\,,\\ \mathrm{Tan}^{-1}\frac{F^2}{|H_1^+|^2} &\equiv \alpha_Q\,,\ H_{1F}^+ = \frac{H_1^+}{F}\,. \end{aligned}\right\} \tag{3.110}$$

Then the solution of (3.107) and (3.108) can be expressed in the form

$$\frac{X_1 + \ell_w X_3}{a} = \frac{1}{KF}\frac{H_{1F}^+}{\big\{\,1 + i|H_{1F}^+|^2\big\}} = \frac{|\cos\alpha_Q|}{iK\overline{H}_1^+}\,e^{i\alpha_Q} \tag{3.111}$$

$$= \frac{|\cos\alpha_Q|}{\bar{A}_1}\,e^{i\delta_1}\,,\quad \delta_1 = \alpha_Q + \varepsilon_1\,. \tag{3.112}$$

In the antisymmetric motion too, there must be resonance due to the restoring moment in roll, the frequency of which can be given by putting $F = 0$ and thus $\alpha_Q = 0$.

We note that a combined form $X_1 + \ell_w X_3$ of the complex amplitude is important when considering the wave generated by sway and roll, because the antisymmetric radiation wave can be computed as follows:

$$\begin{aligned} \zeta_B^+ &= \zeta_1^+ + \zeta_3^+ = -iKX_1H_1^+ - iKX_3H_3^+\\ &= -iK\big(\,X_1H_1^+ + X_3H_1^+\ell_w\,\big) = -iK\big(\,X_1 + \ell_w X_3\,\big)H_1^+\,. \end{aligned} \tag{3.113}$$

4. Theory of Wave Reflection and Absorption

Based on the knowledge acquired in the preceding chapter, the theory of wave absorption and perfect reflection, which is one of the important subjects in ocean engineering, will be explained. The main part of explanation here is to give a compact formula for the reflection and transmission waves and to understand what is the essential condition for realizing the perfect reflection of incident waves.

4.1 Reflection and Transmission Waves for a Fixed Symmetric Body

The definition of the reflection and transmission waves was already given in section 3.4 for the case of a general-shaped body which freely oscillates in waves. In this section, as the first step to deepen the understanding, let us consider a symmetric body which is fixed in space.

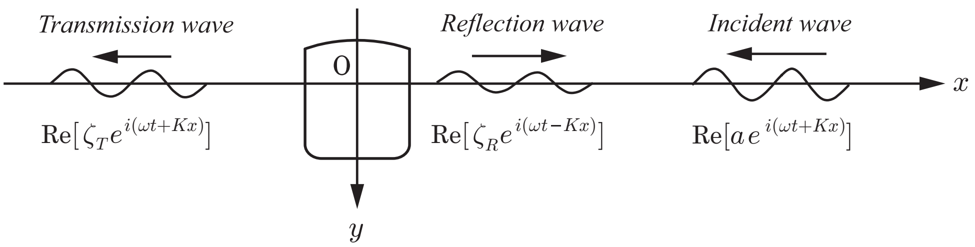


Fig.4.1 Reflection and transmission waves for a symmetric body.

As shown in Fig. 4.1, the incident wave is assumed to be incoming from the positive x-axis. For this case, as given by (3.38) and (3.41), the coefficients of reflection wave R and transmission wave T are defined as

$$R = \frac{\zeta_R}{a} = iH_4^+(K), \tag{4.1}$$

$$T = \frac{\zeta_T}{a} = 1 + iH_4^-(K). \tag{4.2}$$

Since a floating body is symmetric, (3.80) and (3.81) hold. Thus R and T defined above can be expressed with the Kochin functions in the radiation problem as follows:

$$R = \frac{1}{2}\left[\frac{H_2^+}{\overline{H_2^+}} + \frac{H_1^+}{\overline{H_1^+}}\right], \tag{4.3}$$

$$T = \frac{1}{2}\left[\frac{H_2^+}{\overline{H_2^+}} - \frac{H_1^+}{\overline{H_1^+}}\right]. \tag{4.4}$$

It is noteworthy that the first and second terms in brackets represent the symmetric and antisymmetric components, respectively, of the diffraction wave. Although H_1^+ for the antisymmetric component can be replaced with H_3^+, the result can be written eventually only with H_1^+ because of the relation of $H_3^+ = H_1^+\,\ell_w$.

It can be seen from (4.3) and (4.4) that the amplitude of the Kochin function has nothing to do with R and T. To write this explicitly, we use the following expressions for the Kochin function

$$H_1^+ = \frac{i}{K}\bar{A}_1\, e^{i\varepsilon_1}, \quad H_2^+ = \frac{i}{K}\bar{A}_2\, e^{i\varepsilon_2}. \tag{4.5}$$

From these, it follows that

$$\frac{H_1^+}{\overline{H}_1^+} = -\,e^{i2\varepsilon_1}, \quad \frac{H_2^+}{\overline{H}_2^+} = -\,e^{i2\varepsilon_2}. \tag{4.6}$$

Therefore, substituting these in (4.3) and (4.4), we can obtain the following expressions:

$$R = -\frac{1}{2}\left(e^{i2\varepsilon_2} + e^{i2\varepsilon_1}\right) = -\cos(\varepsilon_2 - \varepsilon_1)\,e^{i(\varepsilon_2+\varepsilon_1)}, \tag{4.7}$$

$$T = -\frac{1}{2}\left(e^{i2\varepsilon_2} - e^{i2\varepsilon_1}\right) = -i\sin(\varepsilon_2 - \varepsilon_1)\,e^{i(\varepsilon_2+\varepsilon_1)}. \tag{4.8}$$

It is obvious from these expressions that the relation of energy conservation $|\,R\,|^2 + |\,T\,|^2 = 1$ is satisfied and also the energy equally-splitting law $|\,R \pm T\,| = 1$ is satisfied. These relations were already proven in the preceding chapter in terms of Green's theorem.

4.2 Reflection and Transmission Waves for a Freely Oscillating Symmetric Body

For the case where a symmetric body is freely oscillating in waves, the coefficients of reflection and transmission waves can be computed from (3.37) and (3.40), by superimposing the radiation waves on the diffraction waves R and T expressed by (4.3) and (4.4):

$$C_R = \frac{\zeta_R}{a} = \frac{1}{2}\left[\frac{H_2^+}{\overline{H}_2^+} + \frac{H_1^+}{\overline{H}_1^+}\right] - iK\sum_{j=1}^{3}\left(\frac{X_j}{a}\right)H_j^+, \tag{4.9}$$

$$C_T = \frac{\zeta_T}{a} = \frac{1}{2}\left[\frac{H_2^+}{\overline{H}_2^+} - \frac{H_1^+}{\overline{H}_1^+}\right] - iK\sum_{j=1}^{3}\left(\frac{X_j}{a}\right)(-1)^j H_j^+. \tag{4.10}$$

Here the complex motion amplitude X_j/a must be given as a solution of the equations of body motion.

To understand step by step, let us consider first the case where a body oscillates only in heave. The analytical solution for the heave motion is provided by (3.103). Since the wave generated by heave is symmetric, the symmetric wave component must be modified as follows:

$$\begin{aligned}
\mathcal{A} \equiv \frac{1}{2}\frac{H_2^+}{\overline{H}_2^+} - iK\left(\frac{X_2}{a}\right)H_2^+ &= \frac{1}{2}\frac{H_2^+}{\overline{H}_2^+} - i\,\frac{(H_{2E}^+)^2}{1+i|H_{2E}^+|^2} \\
&= \frac{1}{2}\frac{H_2^+}{\overline{H}_2^+}\left\{\frac{1+i|H_{2E}^+|^2 - 2iH_{2E}^+\overline{H}_{2E}^+}{1+i|H_{2E}^+|^2}\right\} = \frac{1}{2}\frac{H_2^+}{\overline{H}_2^+}\frac{1-i|H_{2E}^+|^2}{1+i|H_{2E}^+|^2}.
\end{aligned} \tag{4.11}$$

Therefore, the complex coefficients of reflection and transmission waves take the following form:

$$C_R = \frac{1}{2}\left[\frac{H_2^+}{\overline{H}_2^+}\frac{1-i|H_{2E}^+|^2}{1+i|H_{2E}^+|^2} + \frac{H_1^+}{\overline{H}_1^+}\right], \tag{4.12}$$

$$C_T = \frac{1}{2}\left[\frac{H_2^+}{\overline{H}_2^+}\frac{1-i|H_{2E}^+|^2}{1+i|H_{2E}^+|^2} - \frac{H_1^+}{\overline{H}_1^+}\right]. \tag{4.13}$$

It can be seen from these equations that even when the motion (heave) is free, the reflection and transmission waves can be computed only with the phase. More specifically, since the phase of $1 + i\left|\,H_{2E}^+\,\right|^2$ is given from (3.102) and (3.104) in the form

$$\mathrm{Tan}^{-1}\left|\,H_{2E}^+\,\right|^2 = \frac{\pi}{2} - \alpha_H = \frac{\pi}{2} + \varepsilon_2 - \delta_2, \tag{4.14}$$

C_R and C_T can be written as follows:

$$\begin{aligned}
C_R &= \frac{1}{2}\left\{-\,e^{i2\varepsilon_2 - i\pi - i2(\varepsilon_2-\delta_2)} - \,e^{i2\varepsilon_1}\right\} \\
&= \frac{1}{2}\left(e^{i2\delta_2} - e^{i2\varepsilon_1}\right) = i\sin(\delta_2 - \varepsilon_1)\,e^{i(\delta_2+\varepsilon_1)},
\end{aligned} \tag{4.15}$$

$$C_T = \frac{1}{2}\left(e^{i2\delta_2} + e^{i2\varepsilon_1}\right) = \cos(\delta_2 - \varepsilon_1)\,e^{i(\delta_2+\varepsilon_1)}. \tag{4.16}$$

From these, as in the case of motion fixed, we can see that the relation of energy conservation $|\,C_R\,|^2 + |\,C_T\,|^2 = 1$ and the energy equally-splitting law $|\,C_R \pm C_T\,| = 1$ are satisfied.

At resonance of heave, $\alpha_H = 0$ thus $\delta_2 = \varepsilon_2$. In this case, we can see from (4.15), (4.16) and (4.7), (4.8) that the following relations hold:

$$\left.\begin{aligned} C_{R,\mathrm{reso}} &= \; i\sin(\varepsilon_2-\varepsilon_1)\,e^{i(\varepsilon_2+\varepsilon_1)} &&= -\,T \\ C_{T,\mathrm{reso}} &= \;\;\;\cos(\varepsilon_2-\varepsilon_1)\,e^{i(\varepsilon_2+\varepsilon_1)} &&= -\,R \end{aligned}\right\} \tag{4.17}$$

Namely, at resonance, the amplitude of reflection wave becomes equal to that of transmission wave for the motion-fixed case, and likewise the amplitude of transmission wave becomes equal to that of reflection wave for the motion-fixed case.

Next, let us fix the heave motion but allow the body motion in sway and roll. In this case, as explained as (3.113), the antisymmetric wave generated by the body motion can be computed by

$$\zeta_B^+ = -iK\big(\,X_1 + \ell_w X_3\,\big)\,H_1^+. \tag{4.18}$$

Thus substituting the analytical result (3.111) for the combined motion of sway and roll in (4.18), we can see that the antisymmetric wave component must be changed in the form

$$\begin{aligned} \mathcal{B} \equiv \frac{1}{2}\frac{H_1^+}{\overline{H_1^+}} - iK\Big\{\Big(\frac{X_1}{a}\Big)H_1^+ + \Big(\frac{X_3}{a}\Big)H_3^+\Big\} &= \frac{1}{2}\frac{H_1^+}{\overline{H_1^+}} - iK\frac{X_1+\ell_w X_3}{a}\,H_1^+ \\ = \frac{1}{2}\frac{H_1^+}{\overline{H_1^+}} - i\frac{(H_{1F}^+)^2}{1+i|H_{1F}^+|^2} &= \frac{1}{2}\frac{H_1^+}{\overline{H_1^+}}\,\frac{1-i|H_{1F}^+|^2}{1+i|H_{1F}^+|^2}. \end{aligned} \tag{4.19}$$

Therefore from (4.9) and (4.10), the reflection and transmission waves can be expressed as

$$C_R = \frac{1}{2}\left[\,\frac{H_2^+}{\overline{H_2^+}} + \frac{H_1^+}{\overline{H_1^+}}\,\frac{1-i|H_{1F}^+|^2}{1+i|H_{1F}^+|^2}\,\right], \tag{4.20}$$

$$C_T = \frac{1}{2}\left[\,\frac{H_2^+}{\overline{H_2^+}} - \frac{H_1^+}{\overline{H_1^+}}\,\frac{1-i|H_{1F}^+|^2}{1+i|H_{1F}^+|^2}\,\right]. \tag{4.21}$$

As shown before for the case of heave only free, these results can be written only with the phase, because the phase of $1 + i\big|\,H_{1F}^+\big|^2$ is given from (3.110) and (3.112) as

$$\mathrm{Tan}^{-1}|H_{1F}^+|^2 = \frac{\pi}{2} - \alpha_Q = \frac{\pi}{2} + \varepsilon_1 - \delta_1.$$

Thus in the same way as in obtaining (4.14) and (4.15), the results may be written as

$$\left.\begin{aligned} C_R &= -\frac{1}{2}\big(\,e^{i2\varepsilon_2} - \,e^{i2\delta_1}\big) = -i\sin(\varepsilon_2-\delta_1)\,e^{i(\varepsilon_2+\delta_1)} \\ C_T &= -\frac{1}{2}\big(\,e^{i2\varepsilon_2} + e^{i2\delta_1}\big) = -\cos(\varepsilon_2-\delta_1)\,e^{i(\varepsilon_2+\delta_1)} \end{aligned}\right\} \tag{4.22}$$

At resonance of sway-roll combined motion, $\alpha_Q = 0$ thus $\delta_1 = \varepsilon_1$. In this case, we can see that

$$\left.\begin{aligned} C_{R,\mathrm{reso}} &= \; -i\sin(\varepsilon_2-\varepsilon_1)\,e^{i(\varepsilon_2+\varepsilon_1)} &&= T \\ C_{T,\mathrm{reso}} &= \; -\cos(\varepsilon_2-\varepsilon_1)\,e^{i(\varepsilon_2+\varepsilon_1)} &&= R \end{aligned}\right\} \tag{4.23}$$

This result is essentially the same as (4.17) for the case of heave only free except for negative sign.

Last, let us consider the general case when all modes of motion are free. In this case, the reflection and transmission waves can be obtained easily from (4.12), (4.13), (4.20), and (4.21) and written in the form

$$C_R = \mathcal{A} + \mathcal{B} = \frac{1}{2}\left[\,\frac{H_2^+}{\overline{H_2^+}}\,\frac{1-i|H_{2E}^+|^2}{1+i|H_{2E}^+|^2} + \frac{H_1^+}{\overline{H_1^+}}\,\frac{1-i|H_{1F}^+|^2}{1+i|H_{1F}^+|^2}\,\right], \tag{4.24}$$

$$C_T = \mathcal{A} - \mathcal{B} = \frac{1}{2}\left[\,\frac{H_2^+}{\overline{H_2^+}}\,\frac{1-i|H_{2E}^+|^2}{1+i|H_{2E}^+|^2} - \frac{H_1^+}{\overline{H_1^+}}\,\frac{1-i|H_{1F}^+|^2}{1+i|H_{1F}^+|^2}\,\right]. \tag{4.25}$$

These results can be rewritten only in terms of the phase of motions as follows:

$$\left.\begin{aligned} C_R &= \frac{1}{2}\left(e^{i2\delta_2} + e^{i2\delta_1}\right) = \cos(\delta_2 - \delta_1)\, e^{i(\delta_2+\delta_1)} \\ C_T &= \frac{1}{2}\left(e^{i2\delta_2} - e^{i2\delta_1}\right) = i\sin(\delta_2 - \delta_1)\, e^{i(\delta_2+\delta_1)} \end{aligned}\right\} \tag{4.26}$$

It can be confirmed again that even in this case where all modes of motion are free, the relation of energy conservation $|\,C_R\,|^2 + |\,C_T\,|^2 = 1$ and the energy equally-splitting law $|\,C_R \pm C_T\,| = 1$ are satisfied.

More importantly, we can see from (4.26) that the perfect reflection and perfect transmission of an incident wave can be realized, if the following relations are satisfied:

$$\left.\begin{aligned} &\text{Perfect reflection :} \quad && \delta_2 - \delta_1 = n\pi \quad (\, n = 0, \pm 1, \cdots) \\ &\text{Perfect transmission :} \quad && \delta_2 - \delta_1 = \frac{\pi}{2} + n\pi \end{aligned}\right\} \tag{4.27}$$

That is to say, the perfect reflection for instance can be realized when the phase difference between the symmetric (heave) and antisymmetric (sway and/or roll) motions is 0 or π, and the amplitude has nothing to do with the conditions for perfect reflection and transmission.

4.3 Wave Drift Force

Up to the preceding section, what we call linear theory has been explained, assuming that both amplitudes of incident wave and resulting unsteady motions of a body are small enough and considering only the first-order terms. The wave drift force to be explained in this section is a time-averaged steady force which is of second order proportional to the square of the incident-wave amplitude. The second-order steady wave force can be computed only in terms of the linear velocity potential, and we need not solve higher-order boundary-value problems. As will be shown later, the wave drift force is directly related to the reflection-wave coefficient already studied.

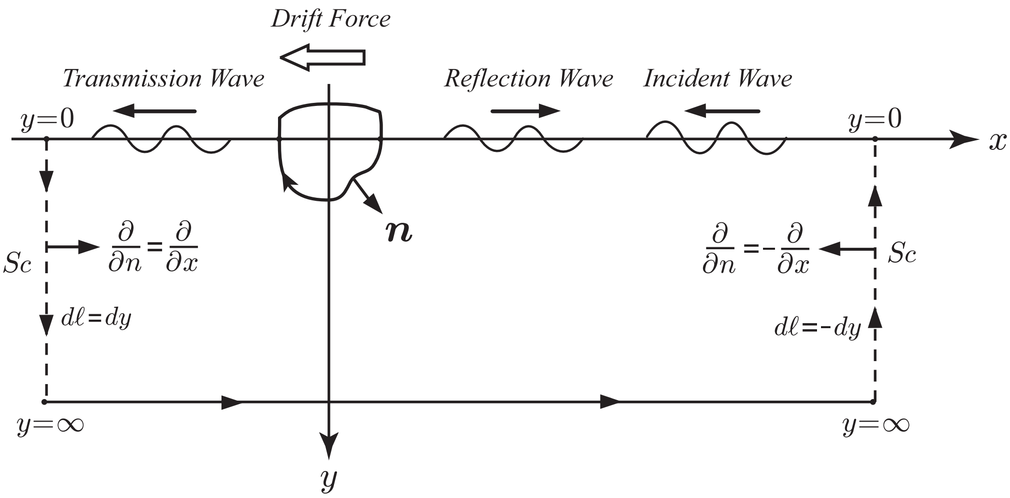


Fig. 4.2 Analysis for wave drift force.

Now let us apply the principle of momentum conservation to the fluid region shown in Fig. 4.2, bounded by the body surface, free surface, water bottom, and control surfaces (S_c) indicated by dashed lines. In the analysis, the shape of body is not necessarily symmetric with respect to the y-axis, and the incident wave is supposed to be incoming from the positive x-axis. The wave drift force is denoted by F_D and defined as positive when acting in the direction of incident-wave propagation.

【Note】 Momentum Conservation Principle

Derivation of the equations to be obtained from the principle of momentum conservation can be similar to that shown for the principle of energy conservation in Section 1.4. Since the momentum is a vector quantity, let us denote its i-th component as M_i. Its rate of change with respect to time must be considered in a Lagrangian way; that is, the transport theorem should be applied. Considering the general 3D problem, we can write as follows:

$$\frac{dM_i}{dt} = \frac{d}{dt}\iiint_{V(t)} \rho u_i \, dV = \rho \iiint_V \frac{\partial u_i}{\partial t} \, dV + \rho \iint_S u_i \, U_n \, dS, \tag{4.28}$$

where u_i is the i-th component of the fluid velocity, and U_n is the normal velocity of the boundary surface S, pointing out of the fluid volume under consideration.

Using the continuity equation (1.11) and Euler's equation (1.14), it follows that

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho} - gz\right) - \frac{\partial}{\partial x_j}\left(u_j \, u_i \right). \tag{4.29}$$

Substituting this into (4.28) and applying Gauss' theorem, we have

$$\frac{dM_i}{dt} = -\rho \iint_S \left[\frac{p}{\rho} n_i + u_i \big(u_n - U_n \big) \right] dS. \tag{4.30}$$

Here we note that the contribution of gz in (4.29) is zero, because only the horizontal components $(i = 1, 2)$ are considered.

The boundary surface encompassing the fluid volume consists of the body surface (S_H), free surface (S_F), and a control surface (S_C) which is at rest and located far from the body. On these boundaries,

$$\left.\begin{array}{ll} \text{on } S_C & U_n = 0 \\ \text{on } S_H & u_n = V_n = U_n \\ \text{on } S_F & u_n = U_n, \; p = 0 \end{array}\right\} \tag{4.31}$$

must be satisfied. (The reference value of the pressure is taken as the atmospheric pressure.) Therefore it follows that

$$\frac{dM_i}{dt} = -\iint_{S_H} p \, n_i \, dS - \iint_{S_C} \big[p \, n_i + \rho u_i u_n \big] \, dS. \tag{4.32}$$

Now let us consider the time average of the above over one cycle. Considering the entire fluid, the time average of the left-hand side (the rate of change in time of the total momentum) must be zero, because the time harmonic oscillation is assumed. Noting that the positive normal vector is directed outward from the fluid region which is opposite to that in Chapter 3, we can see that the first term on the right-hand side of (4.32) is negative of the force acting in the i-th direction on the body. Thus we can write as follows:

$$\overline{F_i} = \overline{\iint_{S_H} p \, n_i \, dS} = -\overline{\iint_{S_C} \big[p n_i + \rho u_i u_n \big] \, dS}. \tag{4.33}$$

This relation tells us that the force on a floating body can be obtained from an integral on the control surface far from the body. In general, the flow field near the body is complicated, whereas on the control surface, local disturbances decay and only the progressive wave components remain, and hence the analysis on the control surface becomes much simpler. This idea is referred to as the far-field method and will be used for the analysis of the wave drift force.

The analysis for the wave drift force is based on the momentum-conservation principle which is explained in **Note** above. Considering the 2-D case of the analysis, we can obtain the following:

$$F_D = -\overline{\int_{S_C} \left[\, p\, n_x + \rho\, u_x u_n \,\right] d\ell} \tag{4.34}$$

where p denotes the pressure, and u_x and u_n denote the x- and normal components of the fluid velocity, respectively.

$$p = -\rho \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi - gy \right] \tag{4.35}$$

$$u_x\, u_n = \left(\frac{\partial \Phi}{\partial x} \right)^2 n_x, \quad \eta_{\pm\infty} = \frac{1}{g} \frac{\partial \Phi}{\partial t} \bigg|_{y=0,\, x=\pm\infty} \tag{4.36}$$

Taking account of the above, substituting these into (4.34), and retaining the terms up to $O(\Phi^2)$, we have

$$\begin{aligned} F_D &= \rho \overline{\left[\int_0^{\eta_\infty} - \int_0^{\eta_{-\infty}} \right] \left(\frac{\partial \Phi}{\partial t} - gy \right) dy} + \frac{\rho}{2} \int_0^\infty dy \left[\overline{\left(\frac{\partial \Phi}{\partial x} \right)^2 - \left(\frac{\partial \Phi}{\partial y} \right)^2} \right]_{x=-\infty}^{x=+\infty} \\ &= \frac{1}{2}\, \rho g \left(\overline{\eta_\infty^2} - \overline{\eta_{-\infty}^2} \right) + \frac{\rho}{2} \int_0^\infty dy \left[\overline{\left(\frac{\partial \Phi}{\partial x} \right)^2 - \left(\frac{\partial \Phi}{\partial y} \right)^2} \right]_{x=-\infty}^{x=+\infty} \end{aligned} \tag{4.37}$$

Here we consider time harmonic motions and hence the quantities are written in the form

$$\left. \begin{aligned} &\Phi(x,y,t) = \mathrm{Re}\left\{ \phi(x,y)\, e^{i\omega t} \right\} \\ &\eta_{\pm\infty} = \mathrm{Re}\left\{ a_{\pm\infty}\, e^{i\omega t} \right\} \end{aligned} \right\} \tag{4.38}$$

with time-dependent part expressed as $e^{i\omega t}$.

Performing the calculation of time average in terms of (1.75), (4.37) can be reduced to

$$\begin{aligned} F_D =& \frac{1}{4}\, \rho g\ \mathrm{Re}\left\{ a_\infty a_\infty^* - a_{-\infty} a_{-\infty}^* \right\} \\ &+ \frac{\rho}{4}\ \mathrm{Re} \int_0^\infty dy \left[\frac{\partial \phi}{\partial x} \frac{\partial \phi^*}{\partial x} - \frac{\partial \phi}{\partial y} \frac{\partial \phi^*}{\partial y} \right]_{x=-\infty}^{x=+\infty} \end{aligned} \tag{4.39}$$

where we note that there exist the incident and reflected waves at $x = +\infty$ and the transmitted wave at $x = -\infty$. Thus we write as follows:

$$\text{at} \quad x = +\infty \qquad a_\infty = a\, e^{iKx} + \zeta_R\, e^{-iKx}\,, \quad \phi = \frac{g}{i\omega}\, e^{-Ky}\ a_\infty \tag{4.40}$$

$$\text{at} \quad x = -\infty \qquad a_{-\infty} = \zeta_T\, e^{iKx}\,, \quad \phi = \frac{g}{i\omega}\, e^{-Ky}\ a_{-\infty} \tag{4.41}$$

Here we note that both ζ_T and ζ_R should be regarded as complex.

Necessary calculations at $x = +\infty$ are

$$\begin{aligned} &\frac{\partial \phi}{\partial x} = \omega\, e^{-Ky} \left(a\, e^{iKx} - \zeta_R\, e^{-iKx} \right), \quad \frac{\partial \phi}{\partial y} = i\omega\, e^{-Ky} \left(a\, e^{iKx} + \zeta_R\, e^{-iKx} \right) \\ &a_\infty\, a_\infty^* = \left(a\, e^{iKx} + \zeta_R\, e^{-iKx} \right)\left(a\, e^{-iKx} + \zeta_R^*\, e^{-iKx} \right) \\ &\qquad = a^2 + |\zeta_R|^2 + 2\, \mathrm{Re}\left(a\, \zeta_R\, e^{-i2Kx} \right) \end{aligned}$$

Therefore the result takes the following form:

$$\begin{aligned} F_D^\infty &\equiv \frac{1}{4}\, \rho g\, \mathrm{Re}\left(a_\infty\, a_\infty^* \right) + \frac{1}{4}\, \rho\, \mathrm{Re} \int_0^\infty dy \left[\frac{\partial \phi}{\partial x} \frac{\partial \phi^*}{\partial x} - \frac{\partial \phi}{\partial y} \frac{\partial \phi^*}{\partial y} \right]_{x=+\infty} \\ &= \frac{1}{4}\, \rho g \left(a^2 + |\zeta_R|^2 \right) + \frac{1}{2}\, \rho g\, \mathrm{Re}\left(a\, \zeta_R\, e^{-i2Kx} \right) \\ &\quad + \frac{1}{4}\, \rho g\ \frac{1}{2} \left[a^2 + |\zeta_R|^2 - 2\, \mathrm{Re}\left(a\, \zeta_R\, e^{-i2Kx} \right) - \left\{ a^2 + |\zeta_R|^2 + 2\, \mathrm{Re}\left(a\, \zeta_R\, e^{-i2Kx} \right) \right\} \right] \\ &= \frac{1}{4}\, \rho g \left(a^2 + |\zeta_R|^2 \right) \end{aligned} \tag{4.42}$$

In the same way, calculations at $x = -\infty$ are

$$\frac{\partial \phi}{\partial x} = \omega\, e^{-Ky}\, \zeta_T\, e^{iKx}\,, \quad \frac{\partial \phi}{\partial y} = i\omega\, e^{-Ky}\, \zeta_T\, e^{iKx}\,, \quad a_{-\infty}\, a_{-\infty}^{*} = |\zeta_T|^2$$

Thus

$$\begin{aligned} F_D^{-\infty} &\equiv \frac{1}{4}\,\rho g\, \mathrm{Re}\big(\, a_{-\infty}\, a_{-\infty}^{*}\,\big) + \frac{\rho}{4}\,\mathrm{Re}\int_0^{\infty} dy\, \left[\, \frac{\partial \phi}{\partial x}\frac{\partial \phi^*}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial \phi^*}{\partial y}\,\right]_{x=-\infty} \\ &= \frac{1}{4}\,\rho g\, |\zeta_T|^2 + \frac{1}{4}\,\rho g\, \frac{1}{2}\,\{\, |\zeta_T|^2 - |\zeta_T|^2\,\} \\ &= \frac{1}{4}\,\rho g |\zeta_T|^2 \end{aligned} \tag{4.43}$$

Substituting these into (4.39) gives the following result:

$$\begin{aligned} F_D = F_D^{\infty} - F_D^{-\infty} &= \frac{1}{4}\,\rho g\,\big(\, a^2 + |\zeta_R|^2 - |\zeta_T|^2\,\big) \\ &= \frac{1}{4}\,\rho g a^2 \big(\, 1 + |C_R|^2 - |C_T|^2\,\big) \end{aligned} \tag{4.44}$$

On the other hand, the relation of energy conservation $|\, C_R|^2 + |\, C_T|^2 = 1$ is satisfied even when the body motion is free to respond in waves. Thus substituting this relation into the above, we can obtain finally the calculation formula for the wave drift force in the form

$$\frac{F_D}{\frac{1}{2}\rho g a^2} \equiv F_D' = |C_R|^2 \tag{4.45}$$

We can see from this result that the normalized wave-drift force can be computed with square of the reflection-wave coefficient; which is positive, implying that a floating body may drift to the downwave side (in the same direction of incident-wave propagation) while reflecting the incident wave.

Suppose that a floating-type breakwater is designed such that a larger part of the incident wave will be reflected with little transmission of the wave. This floating breakwater is efficient in its performance, but we should realize that the more the performance is efficient, the larger the wave-drift force acts, and consequently the tension force acting on a mooring line becomes larger, as implied from (4.45),

By the way, in the analysis above, the principle of energy conservation was applied in the transformation from (4.44) to (4.45). However, as will be studied in the next section, in the presence of energy dissipation (for instance when the absorption of wave energy is made by an exterior mechanical system), what form of the energy relation should be? In order to answer this question, let us consider the energy relation again by denoting the energy dissipation as $\varDelta E$. First we start with the equation from (1.38)

$$\begin{aligned} \varDelta E &= -\rho \left[\, \int_{S_\infty} + \int_{S_{-\infty}} \,\right] \overline{\frac{\partial \varPhi}{\partial t}\frac{\partial \varPhi}{\partial n}}\, d\ell \\ &= -\,\frac{1}{4}\,\rho\omega\, \mathrm{Im}\int_0^{\infty} dy\, \left[\, \phi\frac{\partial \phi^*}{\partial x} - \phi^*\frac{\partial \phi}{\partial x}\,\right]_{x=-\infty}^{x=+\infty} \end{aligned} \tag{4.46}$$

Then, since we have

$$\left.\begin{aligned} \phi &= \frac{g}{i\omega}\, e^{-Ky}\big(\, a\, e^{iKx} + \zeta_R\, e^{-iKx}\big) \\ \frac{\partial \phi^*}{\partial x} &= \omega\, e^{-Ky}\big(\, a\, e^{-iKx} - \zeta_R^*\, e^{iKx}\big) \end{aligned}\right\} \quad \text{as } x \to +\infty \tag{4.47}$$

$$\left.\begin{aligned} \phi &= \frac{g}{i\omega}\, e^{-Ky}\zeta_T\, e^{iKx} \\ \frac{\partial \phi^*}{\partial x} &= \omega\, e^{-Ky}\, \zeta_T^*\, e^{-iKx} \end{aligned}\right\} \quad \text{as } x \to -\infty \tag{4.48}$$

we can substitute these results into (4.46), providing the following relation:

$$\begin{aligned} \Delta E &= \frac{1}{4}\,\frac{\rho g \omega}{K}\left(a^2 - |\,\zeta_R\,|^2 - |\,\zeta_T\,|^2\right) \\ &= \frac{1}{4}\,\rho g a^2 \frac{\omega}{K}\left(1 - |\,C_R\,|^2 - |\,C_T\,|^2\right) \end{aligned} \tag{4.49}$$

We can see naturally from this result that the well-known energy conservation, $|\,C_R\,|^2 + |\,C_T\,|^2 = 1$, can be obtained for the case of $\Delta E = 0$. On the other hand, if $\Delta E \neq 0$, in terms of the energy loss η defined by

$$\eta \equiv \frac{\Delta E}{E_W} = 1 - |\,C_R\,|^2 - |\,C_T\,|^2, \qquad \left(E_W = \frac{1}{4}\,\rho g a^2 \frac{\omega}{K}\right) \tag{4.50}$$

we can have a generalized expression after substituting the above relation into (4.44), in the form

$$\frac{F_D}{\frac{1}{2}\,\rho g a^2} = F_D' = |\,C_R\,|^2 + \frac{1}{2}\,\eta \tag{4.51}$$

According to this formula, we can see that the wave drift force can not be zero, even if the complete absorption of wave energy can be realized.

4.4 Theory for Wave Absorption by a Symmetric Floating Body

The coefficients of reflection and transmission waves for the case when a symmetric body is freely oscillating in waves can be computed from (4.9) and (4.10). In this section, we consider how both reflection and transmission waves can be completely zero by using an active control for the complex amplitude of body motion.

For brevity in the analysis, instead of (4.9) and (4.10), we separate the wave into symmetric and anti-symmetric components, as shown in (3.100). The result takes the form

$$\text{Symmetric wave} = \frac{1}{2}\left(C_R + C_T\right) = \frac{1}{2}\,\frac{H_2^+}{\overline{H}_2^+} - iK\left(\frac{Y}{a}\right)H_2^+ \tag{4.52}$$

$$\text{Anti-symmetric wave} = \frac{1}{2}\left(C_R - C_T\right) = \frac{1}{2}\,\frac{H_1^+}{\overline{H}_1^+} - iK\left(\frac{X_G + \ell_n \Theta}{a}\right)H_1^+ \tag{4.53}$$

Here the result of (4.18) has been used in (4.53).

Since making both reflection and transmission waves zero is equivalent to making both symmetric and anti-symmetric waves zero, we can see from (4.52) and (4.53) that the complex amplitude of body motions must be of the following values:

$$Y = \frac{1}{2}\,\frac{a}{iK\overline{H}_2^+} = \frac{1}{2}\,\frac{a}{\bar{A}_H}\,e^{i\varepsilon_H} \tag{4.54}$$

$$X_G + \ell_n \Theta = \frac{1}{2}\,\frac{a}{iK\overline{H}_1^+} = \frac{1}{2}\,\frac{a}{\bar{A}_S}\,e^{i\varepsilon_S} \tag{4.55}$$

We can recognize that (4.54) is just half the complex amplitude of heave at resonance. Similarly, (4.55) is half the complex amplitude of anti-symmetric motion at its resonance.

Next, let us consider how the conditions (4.54) and (4.55) for perfect wave absorption can be realized in a practical situation. First we consider (4.54) for the problem of symmetric motion.

As depicted in the left of Fig. 4.3, we introduce an exterior mechanical system consisting of mass, dashpot, and spring. Denoting the coefficients in the inertia force, damping force, and restoring force

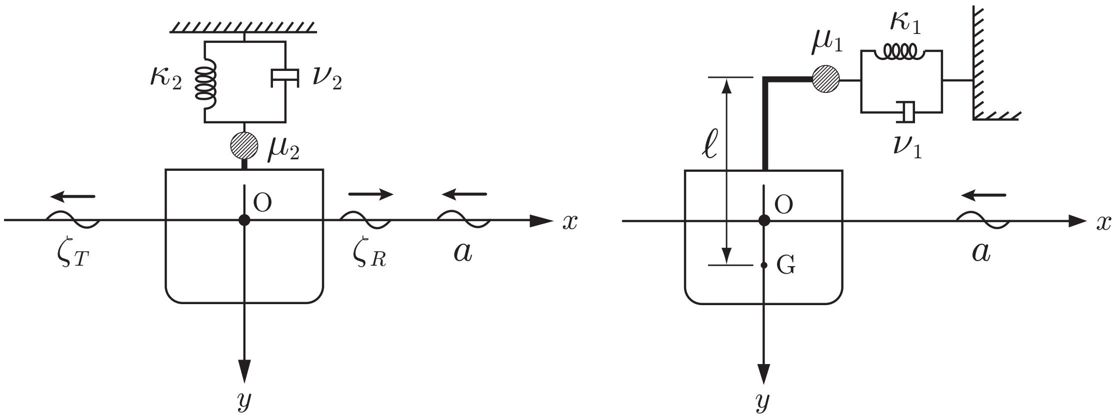


Fig. 4.3 Wave absorption problem by a symmetric floating body.

with μ_2, ν_2, and κ_2, respectively, the motion equations of a floating body and an exterior mechanical system may be expressed as follows:

$$\left.\begin{aligned} \mu_2\,\ddot{y} + \nu_2\,\dot{y} + \kappa_2\,y &= -R \\ (m + A_{22})\,\ddot{y} + B_{22}\,\dot{y} + \rho g B\,y &= E_2\,e^{i\omega t} + R \end{aligned}\right\} \tag{4.56}$$

Therefore, by eliminating an internal force denoted as R, it follows that

$$\big(\,m + A_{22} + \mu_2\,\big)\,\ddot{y} + \big(\,B_{22} + \nu_2\,\big)\,\dot{y} + \big(\,\rho g B + \kappa_2\,\big)\,y = E_2\,e^{i\omega t} \tag{4.57}$$

where we will introduce the following relations

$$B_{22} = \rho\omega|H_2^+|^2, \quad E_2 = \rho g a H_2^+$$

and notations for making equations compact

$$\left.\begin{aligned} \nu_2 \equiv \beta_2\,B_{22} = \beta_2\,\rho\omega|H_2^+|^2 \\ \rho g B + \kappa_2 - \omega^2\big(m + A_{22} + \mu_2\big) \equiv \rho\omega^2 E^2 \end{aligned}\right\} \tag{4.58}$$

Then in a similar way to deriving (3.104) and (3.112), the complex amplitude of heave motion Y in the expression of $y = Ye^{i\omega t}$ can be given in the following form:

$$KY = \frac{a\ H_2^+}{E^2 + i(1+\beta_2)|H_2^+|^2} \tag{4.59}$$

In order for the complex amplitude given by the above result to be equal to (4.54), we can see that $E = 0$ and $\beta_2 = 1$. Namely, as instructed from (4.58), we should tune the values of κ_2 and μ_2 to make the heave motion be resonant and tune the value of ν_2 (exterior damping coefficient) in a way that ν_2 becomes equal to the wave-making damping coefficient of the floating body.

This condition for perfect wave absorption may be expressed in a different way. First we introduce "mechanical impedance" for each of the motion equations as follows:

$$\left.\begin{aligned} Z_{22} &\equiv B_{22} + \frac{1}{i\omega}\big\{\rho g B - \omega^2(m + A_{22})\big\} \\ Z_2^e &\equiv \nu_2 + \frac{1}{i\omega}\big\{\kappa_2 - \omega^2\mu_2\big\} \end{aligned}\right\} \tag{4.60}$$

Then the conditions of $E = 0$ and $\beta_2 = 1$ (i.e. $B_{22} = \nu_2$) are equivalent to the following condition:

$$\overline{Z}_{22} = Z_2^e \tag{4.61}$$

where the overbar means the complex conjugate. This relation is known as the "conjugate matching" of the impedance. By the way, the motion equation corresponding to (4.59) may be rewritten with the mechanical impedance in the following form:

$$\big(\, Z_{22} + Z_2^e \,\big)\, i\omega Y = E_2 \tag{4.62}$$

Up to this point, we could absorb the wave energy of symmetric component around a symmetric floating body with the condition of (4.54). This wave energy must be equal to the work done by the exterior mechanical system. To see the amount of absorbed energy, let us calculate the rate of work done by the exterior damping force. Using (4.59) and (4.61), we can obtain the following result:

$$W_2 = \frac{1}{T}\int_0^T \nu_2\, \dot{y}\, dy = \frac{1}{2}\,\nu_2\,\omega^2 |Y|^2 = \frac{1}{2}\left(\frac{g}{\omega}\right)^2 \nu_2\, |KY|^2 = \frac{1}{2}\,\nu_2\, \frac{|E_2|^2}{|Z_{22}+Z_2^e|^2} \tag{4.63}$$

$$= E_W \frac{2\beta_2 |H_2^+|^4}{|E^2 + i(1+\beta_2)|H_2^+|^2|^2}, \tag{4.64}$$

where

$$E_W = \frac{1}{2}\rho g a^2\, c_g = \frac{1}{2}\rho g a^2 \frac{g}{2\omega} = \frac{1}{4}\frac{\rho g^2 a^2}{\omega}. \tag{4.65}$$

This quantity E_W is the mean rate of energy flux of incident wave per unit area on the mean free surface, see (1.76). For the case of $E = 0$ and $\beta_2 = 1$, the rate of work given by (4.64) reduces to the following

$$W_2 = \frac{1}{2} E_W \tag{4.66}$$

Namely, half of the incident-wave energy per unit time can be absorbed by controlling the symmetric motion of a symmetric body. The remaining half is expected to be absorbed by controlling the anti-symmetric motion of a symmetric body, which will be shown in what follows.

Before going further, however, a coupled of notes should be mentioned. First, by considering (4.64) as a function of β_2 (where $\beta_2 > 0$), its maximum can be achieved at $E = 0$ and $\beta_2 = 1$; that is, when the perfect wave absorption is realized, the amount of absorbed wave energy is at its maximum. Second, (4.63) expressed with the mechanical impedance is valid also for 3-D problems, not necessarily limited to 2-D problems.

Now let us consider the case of the right in Fig. 4.3, with a horizontal exterior mechanical system applied. The reaction force from the exterior system is set to act at a point with moment lever ℓ from the center of gravity G. Then the coupled motion equations of the exterior mechanical system and anti-symmetric body motions (sway and roll) may be expressed in the form

$$\left.\begin{aligned} &\mu_1\big(\ddot{x}_G + \ell\ddot{\phi}\big) + \nu_1\big(\dot{x}_G + \ell\dot{\phi}\big) + \kappa_1\big(x_G + \ell\phi\big) = -R_1 \\ &\big(m + A_{11}\big)\,\ddot{x}_G + B_{11}\,\dot{x}_G \qquad\qquad + A_{11}\ell_m\ddot{\phi} + B_{11}\ell_n\dot{\phi} = \rho g a H_1^+ e^{i\omega t} + R_1 \\ &\big(I + I_R\big)\,\ddot{\phi} + B_{11}\ell_n^2\,\dot{\phi} + W\,\overline{GM}\,\phi + A_{11}\ell_m\,\ddot{x}_G + B_{11}\ell_n\,\dot{x}_G = \rho g a H_1^+ \ell_n\, e^{i\omega t} + R_1\ell \end{aligned}\right\} \tag{4.67}$$

Eliminating the reaction force from these, we can obtain the following:

$$\begin{aligned} &\big(\,m + A_{11} + \mu_1\big)\,\ddot{x}_G + \big(B_{11} + \nu_1\big)\,\dot{x}_G + \kappa_1\, x_G \\ &\qquad + \big(A_{11}\ell_m + \mu_1\ell\,\big)\,\ddot{\phi} + \big(B_{11}\ell_n + \nu_1\ell\,\big)\,\dot{\phi} + \kappa_1\ell\,\phi = \rho g a H_1^+ e^{i\omega t} \end{aligned} \tag{4.68}$$

$$\begin{aligned} &\big(\,I + I_R + \mu_1\ell^2\big)\,\ddot{\phi} + \big(B_{11}\ell_n^2 + \nu_1\ell^2\,\big)\,\dot{\phi} + \big(W\overline{GM} + \kappa_1\ell^2\,\big)\,\phi \\ &\qquad + \big(A_{11}\ell_m + \mu_1\ell\,\big)\,\ddot{x}_G + \big(B_{11}\ell_n + \nu_1\ell\,\big)\,\dot{x}_G + \kappa_1\ell\, x_G = \rho g a H_1^+ \ell_n\, e^{i\omega t} \end{aligned} \tag{4.69}$$

Here we introduce the following notations for simplifying the results

$$\left.\begin{array}{l}\nu_1 \equiv \beta_1 \; B_{11} = \beta_1 \; \rho\omega |H_1^+|^2 \\ \kappa_1 - \omega^2 (m + A_{11} + \mu_1) \equiv \rho\omega^2 S^2 \\ \kappa_1 \ell - \omega^2 (A_{11}\ell_m + \mu_1 \ell\,) \equiv \rho\omega^2 Q^2 \ell_n \\ \kappa_1 \ell^2 + W\overline{GM} - \omega^2 (I + I_R + \mu_1 \ell^2) \equiv \rho\omega^2 R^2 \ell_n^2 \\ F^2 = (S^2 R^2 - Q^4)/(S^2 + R^2 - 2Q^2) \end{array}\right\} \tag{4.70}$$

Then the same transformation as for obtaining (3.111) provides the following compact expression:

$$K\big(\, X_G + \ell_n \Theta \,\big) = \frac{a\, H_1^+}{F^2 + i(1 + \beta_1 \ell/\ell_n)|H_1^+|^2} \tag{4.71}$$

We can see from this result that (4.55) can be realized with $F = 0$ and $\beta_1 \ell/\ell_n = 1$. In this particular case, the rate of work to be done by the exterior damping force can be found to be

$$W_1 = E_W \frac{2\beta_1 |H_1^+|^4}{|F^2 + i(1 + \beta_1 \ell/\ell_n)|H_1^+|^2|^2} \longrightarrow \frac{1}{2}\, E_W \tag{4.72}$$

Therefore, summing up (4.66) and (4.72), we have

$$(W_1 + W_2)_{\max} = E_W \tag{4.73}$$

This means that, as expected, the total wave energy could be absorbed by controlling both symmetric and anti-symmetric motions of a symmetric body independently.

4.5 Wave Absorption by a One-side Waveless Body

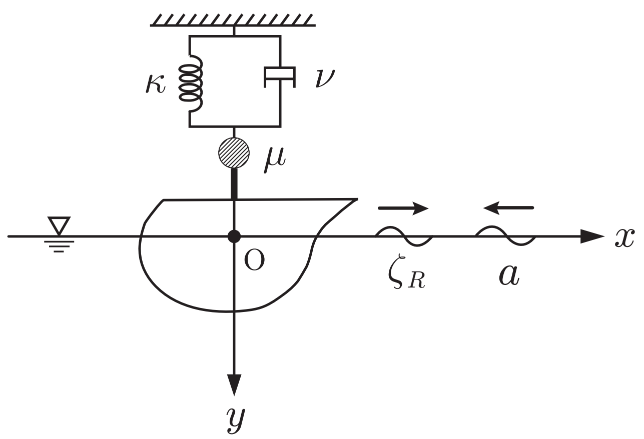


Fig. 4.4 Wave absorption by a one-side waveless body.

The independent optimal control described above is actually cumbersome in practice. Thus in the present section, we will consider an asymmetric floating body, especially the so-called "one-side waveless body" which generates no wave at all in one direction even when a body is forcedly oscillated.

For simplicity, let us analyze the heave motion of this waveless body, with an exterior mechanical system applied as shown in Fig. 4.4. As before, the coefficients of inertia force, damping force, and restoring force are denoted as μ, ν, and κ, respectively. Then the motion equation of this floating body can be written in a form similar to (4.57) as follows:

$$\big(\, m + A_{22} + \mu \,\big)\, \ddot{y} + \big(\, B_{22} + \nu \,\big)\, \dot{y} + \big(\, \rho g B + \kappa \,\big)\, y = E_2\, e^{i\omega t} \tag{4.74}$$

Let us assume and consider a waveless body which generates no wave in the negative x-axis (downwave side). In this case $H_2^-(K) = 0$ and hence the heave damping coefficient of the body and the wave-exciting force in heave are expressed from (3.61) and (3.75) respectively, in the form

$$B_{22} = \frac{1}{2}\rho\omega\Big\{ |H_2^+|^2 + |H_2^-|^2 \Big\} = \frac{1}{2}\rho\omega\,|H_2^+|^2 \tag{4.75}$$

$$E_2 = \rho g a H_2^+ \tag{4.76}$$

Furthermore, since the relation of (3.79) takes the following form for the present case

$$H_2^+ = \overline{H}_2^+ R + \overline{H}_2^- T = \overline{H}_2^+ R, \tag{4.77}$$

it is obvious that $|R| = 1$ and hence $|T| = 0$ from the principle of energy conservation (3.70). Namely even when the motions are fixed, a one-side waveless body does not transmit an incident wave (not generate the wave in the negative x-axis). Taking account of this fact, when the heave motion of this waveless body is free in waves, the coefficients of reflection and transmission waves can be written from (3.37) and (3.40) respectively, in the following form:

$$C_R = R - iK\Big(\frac{Y}{a}\Big)H_2^+ = \frac{H_2^+}{\overline{H}_2^+} - iK\Big(\frac{Y}{a}\Big)H_2^+ \tag{4.78}$$

$$C_T = T - iK\Big(\frac{Y}{a}\Big)H_2^- = 0 \tag{4.79}$$

Substituting (4.75) and (4.76) in (4.74) and defining the notations of β and E by

$$\left.\begin{aligned} &\nu \equiv \beta\, B_{22} = \beta\,\frac{1}{2}\rho\omega\,|H_2^+|^2 \\ &\rho g B + \kappa - \omega^2\big(m + A_{22} + \mu\big) \equiv \rho\omega^2 E^2 \end{aligned}\right\} \tag{4.80}$$

the complex amplitude Y in the expression of heave motion $y = Ye^{i\omega t}$ can be given in the form

$$KY = \frac{a\,H_2^+}{E^2 + i(1+\beta)\frac{1}{2}|H_2^+|^2} \tag{4.81}$$

Thus substituting this result into (4.78) gives the following result

$$C_R = \frac{H_2^+}{\overline{H}_2^+} - i\,\frac{(H_2^+)^2}{E^2 + i(1+\beta)\frac{1}{2}|H_2^+|^2} \tag{4.82}$$

It can be seen from this result that $C_R = 0$ if $E = 0$ and $\beta = 1$. In this case, $C_T = 0$ from (4.79). Therefore the perfect wave absorption is realized.

In terms of (4.81), let us calculate the work done per unit time (i.e. the absorbed wave power) by the exterior mechanical system; which can be performed by substituting (4.81) in (4.64) and the result reduces to

$$W = \frac{1}{2}\Big(\frac{g}{\omega}\Big)^2 \nu\,|KY|^2 = E_W\,\frac{\beta\,|H_2^+|^4}{|E^2 + i(1+\beta)\frac{1}{2}|H_2^+|^2|^2} \longrightarrow E_W \tag{4.83}$$

This means that, if the body shape is asymmetric and one-side waveless, the perfect absorption of the wave energy can be realized only with a single mode of body motion (heave in the present analysis). This feature is very advantageous from a viewpoint of controlling the body motion. However, the body shape for one-side waveless must vary depending on the wavelength (frequency) of incident wave. Therefore, in order to enhance the efficiency in the wave-energy absorption over a wider spectrum of wave frequencies, we need to devise more.

With that consideration, let us consider next a way for realizing the one-side waveless condition simply by using a symmetric body.

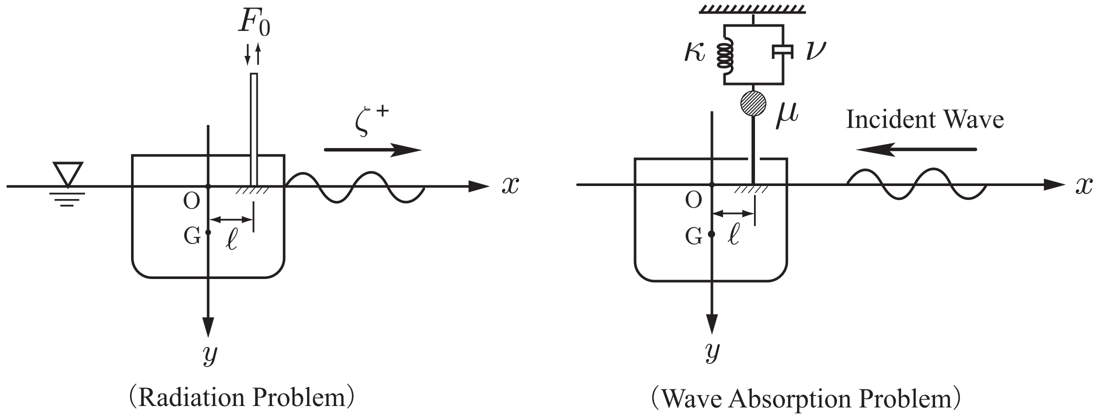


Fig. 4.5 How to realize the condition of one-side waveless using a symmetric body.

4.6 One-side Waveless Condition with a Symmetric Body

As depicted in the left of Fig. 4.5, we consider a radiation (forced oscillation) problem in which a vertical external force F_0 is applied at the point away from the origin with $x_0 = \ell$, which induces the heave and roll motions simultaneously. For a more general case, the position where a vertical external force applies must be (x_0, y_0) and $y_0 \neq 0$. In this case, the analysis may be more complicated, because the origin has the sway velocity. Thus, for simplicity, explanation in the present section will be made with parameter to be determined taken as ℓ only.

By use of mechanical impedance as in (4.62), the coupled motion equations in this special radiation problem can be written in the form

$$\left.\begin{aligned} Z_2\, i\omega Y &= F_0 \\ Z_3\, i\omega \Theta &= F_0\, \ell \end{aligned}\right\} \tag{4.84}$$

Here the mechanical impedances Z_2 and Z_3 are specifically given as follows:

$$\begin{aligned} Z_2 &= B_{22} + \frac{1}{i\omega}\bigl\{\rho g B - \omega^2\bigl(m + A_{22}\bigr)\bigr\} \\ &= \rho\omega\bigl\{\, |H_2^+|^2 - iE^2 \bigr\} = \rho\omega |H_2^+|^2 \sec\alpha_H\, e^{-i\alpha_H} \end{aligned} \tag{4.85}$$

$$\begin{aligned} Z_3 &= B_{33} + \frac{1}{i\omega}\bigl\{\, W\overline{GM} - \omega^2\bigl(I + I_R\bigr)\bigr\} \\ &= \rho\omega\ell_w^2\bigl\{\, |H_1^+|^2 - iR^2 \bigr\} = \rho\omega\ell_w^2\, |H_1^+|^2 \sec\alpha_R\, e^{-i\alpha_R} \end{aligned} \tag{4.86}$$

and the phases α_H and α_R of the impedance are defined as

$$\left.\begin{aligned} &\rho g B - \omega^2\bigl(m + A_{22}\bigr) \equiv \rho\omega^2 E^2, && \tan\alpha_H = \frac{E^2}{|H_2^+|^2} \\ &W\,\overline{GM} - \omega^2\bigl(I + I_R\bigr) \equiv \rho\omega^2\ell_w^2 R^2, && \tan\alpha_R = \frac{R^2}{|H_1^+|^2} \end{aligned}\right\} \tag{4.87}$$

With these preparations, let us calculate the wave elevation propagating in the negative x-axis ζ^-. Since the outgoing radiation wave generated by the forced oscillation can be computed from (3.16) and the modes of motion are heave and roll, we can have the following

$$\begin{aligned} \zeta^- = \zeta_2^- + \zeta_3^- &= iK\bigl\{\, Y H_2^- + \Theta H_3^- \,\bigr\} = -iK\bigl\{\, Y H_2^+ - \Theta \ell_w H_1^+ \,\bigr\} \\ &= -\frac{\omega}{g} F_0 \Bigl[\, \frac{H_2^+}{Z_2} - \ell\, \frac{\ell_w H_1^+}{Z_3} \,\Bigr] \end{aligned} \tag{4.88}$$

Therefore, for realizing the one-side waveless condition (i.e. for the condition of $\zeta^- = 0$), the value of ℓ should be of the following form

$$\ell = \frac{Z_3 H_2^+}{Z_2 \ell_w H_1^+} = \ell_w \frac{\overline{H}_1^+ \sec\alpha_R}{\overline{H}_2^+ \sec\alpha_H}\, e^{i(\alpha_H - \alpha_R)}$$
$$= \ell_w \frac{\bar{A}_S \sec\alpha_R}{\bar{A}_H \sec\alpha_H}\, e^{i(\alpha_H-\alpha_R)+i(\varepsilon_H-\varepsilon_S)} \tag{4.89}$$

Because ℓ must be real, we can see that the condition of one-side waveless must satisfy the following

$$\alpha_H - \alpha_R + \varepsilon_H - \varepsilon_S = n\pi \quad (n = 0, \pm1, \pm2, \cdots) \tag{4.90}$$

This condition is expressed only with the phases of the Kochin function and mechanical impedance, which can also be expressed with the phases of symmetric and anti-symmetric body motions. That is, $\alpha_H + \varepsilon_H = \delta_2$ from (3.104) and $\alpha_R + \varepsilon_S = \delta_1$ from (3.112). Therefore, we can see that (4.90) can be rewritten as follows:

$$\delta_2 - \delta_1 = n\pi \quad (n = 0, \pm1, \pm2, \cdots) \tag{4.91}$$

This relation is the same as (4.27), the condition for perfect reflection for a symmetric body; which implies that we can realize the one-side waveless condition by use of a symmetric floating body which is adjusted to realize the perfect wave reflection!

Once the body shape is given, the Kochin function (i.e. ε_H and ε_S) will be uniquely computed. Thus in order to satisfy (4.90), α_H and α_R should be adjusted; which may be possible in practice by changing the restoring force with mooring lines or by changing E or R defined in (4.87).

When the one-side waveless condition is realized, the complex amplitude of generated wave propagating to the opposite (positive x in the present analysis) direction ζ^+ can be given as follows:

$$\zeta^+ = \zeta_2^+ + \zeta_3^+ = -iK\big\{ Y H_2^+ + \Theta \ell_w H_1^+ \big\}$$
$$= -F_0 \frac{2\, e^{i\alpha_H}}{\rho g \overline{H}_2^+ \sec\alpha_H} = -F_0\, 2i\, \frac{K}{\rho g} \frac{e^{i(\alpha_H+\varepsilon_H)}}{\bar{A}_H \sec\alpha_H} \tag{4.92}$$

From this result, we can see that the vertical external force F_0 should be given in terns of ζ^+ as follows:

$$F_0 = -\frac{1}{2}\rho g \zeta^+ \overline{H}_2^+ \sec\alpha_H\, e^{-i\alpha_H} = i\,\frac{\rho g \zeta^+}{2K}\, \bar{A}_H \sec\alpha_H\, e^{-i(\alpha_H+\varepsilon_H)} \tag{4.93}$$

Now that the condition of one-side waveless could be realized with a symmetric body, the subsequent analysis can be the same as that for an asymmetric one-side waveless body. Namely, as depicted in the right of Fig. 4.5, an external mechanical system should be equipped at the point of $x = \ell$ to be given by (4.89) and (4.90), and then the heave motion should be made resonant by adjusting κ and μ, and the external damping coefficient ν should be made equal to the wave-making damping coefficient of a floating body under consideration. In order to confirm this, let us analyze more specifically the wave absorption problem depicted in Fig. 4.5.

The reaction force between a floating body and an external mechanical system is denoted as F_1. Then the coupled motion equations in waves can be expressed as

$$Z_2\; i\omega Y = \rho g a H_2^+ + F_1 \tag{4.94}$$

$$Z_3\; i\omega \Theta = \rho g a\, \ell_w H_1^+ + F_1\, \ell \tag{4.95}$$

$$\Big[\nu + \frac{1}{i\omega}\big(\kappa - \omega^2 \mu\big)\Big]\, i\omega Y \equiv Z_2^e\; i\omega Y = -F_1 \tag{4.96}$$

where Z_2 and Z_3 are the mechanical impedances given by (4.85) and (4.86), and the wave-exciting force and moment on the right-hand side are expressed with Haskind-Newman's relation.

In order for the symmetric wave component to be perfectly absorbed, the complex amplitude Y in heave must be equal to (4.54). In this case, by calculating the reaction force F_1 from (4.94), we can obtain the following result:

$$F_1 = \frac{ga}{2\omega}\frac{Z_2}{\overline{H}_2^+} - \rho g a H_2^+ = -\rho g a \frac{|H_2^+|^2 + iE^2}{2\overline{H}_2^+}$$
$$= -\frac{1}{2}\rho g a H_2^+ \sec\alpha_H \, e^{i\alpha_H} = -i\,\frac{\rho g a}{2K}\,\bar{A}_H \sec\alpha_H \, e^{i(\alpha_H + \varepsilon_H)} \tag{4.97}$$

We may calculate F_1 from (4.95) in terms of (4.55) for perfect absorption of the anti-symmetric wave component and (4.89)–(4.90) for the one-side waveless condition. Even in this case, we can confirm that the result will be the same as (4.97). Therefore we can conclude that the perfect wave absorption can be achieved by controlling the values of μ, ν, and κ such that F_1 given by (4.96) becomes equal to (4.97).

Those conditions may be confirmed to be the same as the conjugate matching of the mechanical impedance, already given as (4.61). Namely

$$\left.\begin{array}{c} Z_2^e = \overline{Z}_2 \\ \text{that is,}\quad \nu = B_{22} = \rho\omega|H_2^+|^2, \quad \kappa - \omega^2\mu = -\rho\omega^2 E^2 \end{array}\right\} \tag{4.98}$$

When these are satisfied, by use of (4.96) and (4.85) we can easily confirm that the resulting F_1 becomes equal to (4.97).

What should be noted here is that the reaction force F_0 in the forced oscillation for realizing the one-side waveless condition (which is given by (4.93)) is complex conjugate of F_1 given by (4.97). To understand the physical meaning of this relation, first we recall that the time-dependent part was assumed to be $e^{i\omega t}$ and only the real part of the product of spatial (or complex amplitude) and time-dependent parts should be taken. Thus considering the complex conjugate of F_0 is equivalent to write as follows:

$$\mathrm{Re}\left[\,\overline{F}_0\, e^{i\omega t}\,\right] = \mathrm{Re}\left[\,F_0\, e^{-i\omega t}\,\right] = \mathrm{Re}\left[\,F_1\, e^{i\omega t}\,\right]. \tag{4.99}$$

This relation means that reversing the time in the radiation problem is equivalent to considering the wave absorption problem; that is, the reciprocity relation holds in the linear wave theory.

To summarize the above, we need not consider explicitly the diffraction problem and only the information of the radiation problem combined with various hydrodynamic relations suffices for the analysis in the wave-energy absorption problem.

4.7 Revisiting Wave-energy Absorption Theory

In order to make clearer what is noted in the last paragraph of the preceding section, let us revisit the wave absorption problem and analyze in a different manner. First, as explained in connection with the wave drift force in Section 4.3, the energy relation in the case of absorbing the wave energy with certain method is expressed in the form

$$\frac{\varDelta E}{E_W} \equiv \eta = 1 - |\,C_R\,|^2 - |\,C_T\,|^2 \tag{4.100}$$

where

$$E_W = \frac{1}{2}\rho g a^2 \frac{\omega}{2K} = \frac{1}{2}\rho g a^2\, c_g \tag{4.101}$$

is the wave power of incident wave per unit area on the free surface, and hence η defined in (4.100) may be regarded as the efficiency of wave-energy absorption.

By the way, the coefficients of reflection and transmission waves can be computed from (3.37) and (3.40), respectively; these may be expressed as

$$C_R = R - iKX_j H_j^+, \quad C_T = T - iKX_j H_j^- \tag{4.102}$$

Here X_j is meant to be normalized as X_j/a and the summation sign with respect to index j is deleted for simplicity, with the summation convention that any term containing the same index twice should be summed over that index.

In the analysis to follow, we will need the following hydrodynamic relations:

$$\overline{H}_j^+ R + \overline{H}_j^- T = H_j^+ \tag{4.103}$$

$$|R|^2 + |T|^2 = 1 \tag{4.104}$$

These are already proven as (3.79) and (3.70). Then from (4.102) we have the followings:

$$\left.\begin{aligned} |C_R|^2 &= |R|^2 - iKX_j H_j^+ \overline{R} + iK\overline{X}_j \overline{H}_j^+ R + |KX_j H_j^+|^2 \\ |C_T|^2 &= |T|^2 - iKX_j H_j^- \overline{T} + iK\overline{X}_j \overline{H}_j^- T + |KX_j H_j^-|^2 \end{aligned}\right\} \tag{4.105}$$

Substituting these into (4.100) and taking account of (4.103) and (4.104), we have

$$\begin{aligned} \eta &= 2\,\mathrm{Re}\big\{ iKX_j \overline{H}_j^+ \big\} - K^2 |X_j|^2 \big(|H_j^+|^2 + |H_j^-|^2 \big) \\ &\equiv 2\,\mathrm{Re}\,(\gamma) - \frac{|\gamma|^2}{1-\delta} \end{aligned} \tag{4.106}$$

where

$$\gamma \equiv iKX_j \overline{H}_j^+, \quad 1-\delta = \frac{|H_j^+|^2}{|H_j^+|^2 + |H_j^-|^2} > 0 \tag{4.107}$$

Here we note that δ is defined only with the Kochin function and thus can be determined uniquely, once the body shape is given. On the other hand, γ includes the complex amplitude of body motion X_j and the wave absorption efficiency η may vary depending on the value of γ. By viewing (4.106) as a function of γ, we can see that the maximum of η is taken if $\gamma = 1-\delta$. Therefore the maximum of the wave absorption efficiency is given as

$$\eta_{\max} = \gamma_{\max} = 1 - \delta = \frac{|H_j^+|^2}{|H_j^+|^2 + |H_j^-|^2} \tag{4.108}$$

Next, let us consider C_R and C_T when the maximum in the wave absorption efficiency is achieved. Since γ is of real quantity for the case of (4.108), we can transform $|C_R|^2$ in (4.105) in the form

$$|C_R|^2 = |R|^2 - \overline{R}\gamma H_j^+/\overline{H}_j^+ - R\gamma \overline{H}_j^+/H_j^+ + \gamma^2 \tag{4.109}$$

On the other hand, eliminating T from (4.103) and (4.104), we have the following:

$$|R|^2 \Big\{ |H_j^+|^2 + |H_j^-|^2 \Big\} = R\overline{H}_j^{+2} + \overline{R} H_j^{+2} + |H_j^-|^2 - |H_j^+|^2 \tag{4.110}$$

By substituting this relation in (4.109) and using the definition of (4.108), the following result may be obtained

$$|C_R|^2 = 1 - 2\gamma + \gamma^2 = (1-\gamma)^2 = \delta^2 \tag{4.111}$$

$|C_T|$ expressed in a similar way can be obtained by substituting (4.108) and (4.111) into (4.100), and the result takes the form

$$|C_T|^2 = 1 - |C_R|^2 - \eta = \gamma\big(1-\gamma\big) = \big(1-\delta\big)\delta \tag{4.112}$$

Namely
$$\left.\begin{aligned} |\,C_R\,| &= 1-\gamma = \delta \\ |\,C_T\,| &= \sqrt{\gamma(1-\gamma)} = \sqrt{\delta(1-\delta)} \end{aligned}\right\} \tag{4.113}$$

Looking at the results obtained so far in this section, we see that all results are described only with quantities in the radiation problem, irrespective of whether the body is fixed or free to respond in waves.

Considering a one-side waveless body with $H_j^- = 0$, it is obvious from (4.108) that $\eta_{\max} = 1$ is achieved only with a single j-th mode of motion. In this case, $\delta = 0$ and $\gamma = 1$ and hence from (4.107), it follows that

$$X_j = \frac{1}{iK\overline{H}_j^+} \tag{4.114}$$

This result for the complex amplitude of the j-th mode of motion is the same as that at resonance and can be obtained from (4.81) for the case of $E = 0$ and $\beta = 1$. In this case, it is also obvious from (4.113) that $|\,C_R\,| = |\,C_T\,| = 0$, implying that the perfect wave absorption is realized.

Furthermore, for a symmetric body, $H_j^+ = (-1)^j H_j^-$ holds and thus from (4.108) we have $\eta_{\max} = 1/2$, which is the result already explained in Section 4.4. In this case, we can see from (4.113) that $|\,C_R\,| = |\,C_T\,| = 1/2$.

5. Wave Interaction Theory among Multiple Bodies

Wave interactions will be important in the analysis for a catamaran or an offshore structure composed with multiple colums. This chapter is concerned with the so-called wave-interaction theory that can compute interaction effects among multiple bodies only with the diffraction characteristics of elementary bodies. In order to focus on the essence of the theory, explanation in this chapter is limited to 2D problems.

5.1 Diffraction Characteristics of Elementary Bodies

In the wave-interaction theory, all waves reflected by other bodies but incoming to the body under consideration are regarded as incident waves. Thus it is a premise that the diffraction characteristics of each elementary body to "generalized" incident waves (not only propagating but also evanescent components) are known. The calculation method for the diffraction problem was already explained in Chapter 2 but for subsequent convenience, let us summarize it again here.

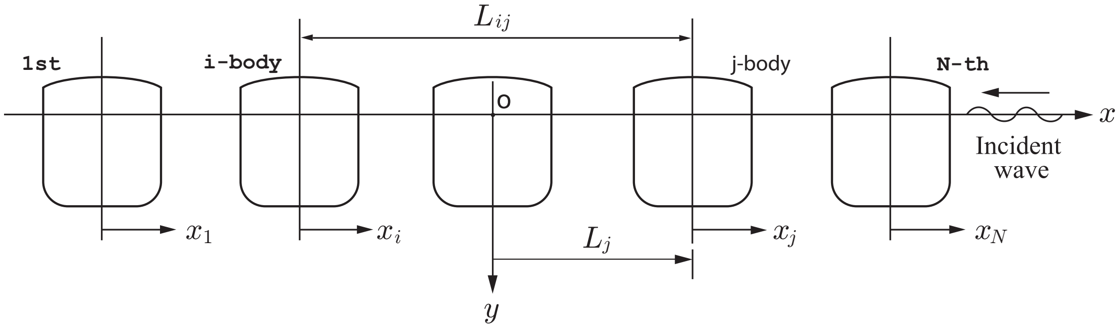


Fig. 5.1 Coordinate system and notations for multiple-body problem.

The number of bodies and the shape of each body can be arbitrary in the theory. However, for simplicity, it is assumed that the shape of elementary bodies is identical and symmetric with respect to the centerline of the body. Thus it is enough to know the diffraction characteristics for one body.

The velocity potential is expressed as

$$\left.\begin{aligned} \Phi(x,y,t) &= \mathrm{Re}\left[\frac{ga}{i\omega}\left\{\phi_I(x,y)+\phi_S(x,y)\right\}e^{i\omega t}\right] \\ \phi_D(x,y) &= \phi_I(x,y)+\phi_S(x,y) \end{aligned}\right\} \tag{5.1}$$

Here ϕ_I denotes the velocity potential of an incident wave incoming from outside, ϕ_S is the scattering potential, and ϕ_D is the sum of those which is referred to as the diffraction potential.

Then the diffraction potential ϕ_D is governed by the Laplace equation and must satisfy appropriate boundary conditions; those are written as follows:

$$[H] \qquad \frac{\partial \phi_D}{\partial n} = 0 \qquad \text{on } S_H \tag{5.2}$$

$$[F] \qquad \frac{\partial \phi_D}{\partial y} + K\phi_D = 0 \qquad \text{on } y=0, \quad K=\frac{\omega^2}{g} \tag{5.3}$$

$$[B] \qquad \frac{\partial \phi_D}{\partial y} = 0 \qquad \text{on } y=h \tag{5.4}$$

where the water depth is denoted as h but it may be treated as $h \to \infty$ afterward.

When applying Green's theorem, we should note that the radiation condition of outgoing waves to be satisfied at infinity (S_∞) is not satisfied by ϕ_D, because ϕ_D includes ϕ_I representing the incoming wave. Thus we obtain the following:

$$C\phi_D(\mathrm{P}) + \int_{S_H} \phi_D(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) = \int_{S_\infty} \left\{ \frac{\partial \phi_D(\mathrm{Q})}{\partial n_\mathrm{Q}} - \phi_D(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} \right\} G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) \tag{5.5}$$

Here C is called the solid angle, equal to 1 when the field point $\mathrm{P}(x,y)$ is located in the fluid region and equal to $1/2$ when $\mathrm{P}(x,y)$ is on a smooth surface of the body. $G(\mathrm{P};\mathrm{Q})$ denotes the free-surface Green function and for the case of finite water depth h, it may be expressed as follows:

$$G(\mathrm{P};\mathrm{Q}) = i\, C_0\, Y_0(y)\, Y_0(\eta)\, e^{-ik_0|x-\xi|} + \sum_{n=1}^{\infty} C_n\, Y_n(y)\, Y_n(\eta)\, e^{-k_n|x-\xi|} \tag{5.6}$$

where

$$\left. \begin{aligned} C_0 &= \frac{k_0}{K + h(k_0^2 - K^2)} \qquad C_n = \frac{k_n}{K - h(k_n^2 + K^2)} \\ Y_0(y) &= \frac{\cosh k_0(y-h)}{\cosh k_0 h}, \quad Y_n(y) = \frac{\cos k_n(y-h)}{\cos k_n h} \end{aligned} \right\} \tag{5.7}$$

$$k_0 \tanh k_0 h = \frac{\omega^2}{g} = K, \quad k_n \tan k_n h = -K \tag{5.8}$$

(5.8) is the dispersion relation for finite water depth, and $k_0 = K$ and $C_0 = 1$ at $h \to \infty$.

Let us consider the right-hand side of (5.5) for ϕ_S and ϕ_I separately. Since ϕ_S satisfies the radiation condition of outgoing wave and ϕ_I has nothing to do with the presence of a body, we have the following relations:

$$0 = \int_{S_\infty} \left\{ \frac{\partial \phi_S(\mathrm{Q})}{\partial n_\mathrm{Q}} - \phi_S(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} \right\} G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) \tag{5.9}$$

$$\phi_I(\mathrm{P}) = \int_{S_\infty} \left\{ \frac{\partial \phi_I(\mathrm{Q})}{\partial n_\mathrm{Q}} - \phi_I(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} \right\} G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) \tag{5.10}$$

Thus, summing up these two, we see that the right-hand side of (5.5) can be expressed simply as ϕ_I and hence we can obtain the following:

$$C\phi_D(\mathrm{P}) + \int_{S_H} \phi_D(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) = \phi_I(\mathrm{P}) \tag{5.11}$$

This equation can be regarded as an integral equation for ϕ_D on the body surface, when the field point $\mathrm{P}(x,y)$ is located on the body surface (i.e. $C = 1/2$). It is important to note that the necessary (satisfying) conditions for ϕ_I on the right-hand side of (5.11) are the Laplace equation and the free-surface boundary condition; that is, the radiation condition needs not to be satisfied and thus evanescent waves can also be included in ϕ_I, not necessarily propagating waves only.

For this kind of "generalized" incident waves, the corresponding scattering potential ϕ_S can be computed and expressed with $C = 1$ in (5.11), in the form

$$\begin{aligned} \phi_S(x,y) &= -\int_{S_H} \phi_D(\mathrm{Q}) \frac{\partial}{\partial n_\mathrm{Q}} G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) \\ &= A_0^{\pm}\, Y_0(y)\, e^{\mp ik_0 x} + \sum_{n=1}^{\infty} A_n^{\pm}\, Y_n(y)\, e^{\mp k_n x} \end{aligned} \tag{5.12}$$

where

$$\left. \begin{aligned} A_0^{\pm} &= -\,i\, C_0 \int_{S_H} \phi_D(\xi,\eta) \frac{\partial}{\partial n} Y_0(\eta)\, e^{\pm ik_0\xi}\, ds(\xi,\eta) \\ A_n^{\pm} &= -\,C_n \int_{S_H} \phi_D(\xi,\eta) \frac{\partial}{\partial n} Y_n(\eta)\, e^{\pm k_n\xi}\, ds(\xi,\eta) \end{aligned} \right\} \tag{5.13}$$

and the double sign ($\pm$) should be taken according as $x > 0$ and $x < 0$, respectively.

if the incident-wave potential ϕ_I is known (explicitly given), coefficients $A_0^{\pm}$ and $A_n^{\pm}$ in (5.13) can be computed directly. However, as in the wave-interaction problem explained here, the complex amplitude of the waves reflected by other bodies is unknown until the problem will be solved, but these reflected waves must be treated as incident waves when viewed from the body concerned. To resolve this situation, we separate the complex amplitude from the other function part of spatial variables and write (5.12) in the following matrix form:

$$\phi_S^i(x_i, y) = \{A_i\}^T \{\psi_S^i(x_i, y)\} \tag{5.14}$$

Here index i is used to denote the i-th body (see Fig. 5.1), and $\{A_i\}^T$ represents a vector consisting of unknown coefficients $\{A_{i0}^+, A_{i0}^-, A_{in}^+, A_{in}^-\}$ $(n = 1, 2, \cdots)$ in the scattering potential. On the other hand, $\{\psi_S^i(x_i, y)\}$ is defined as

$$\{\psi_S^i(x_i, y)\} = \left\{ \begin{array}{ll} u(+x_i)\, Y_0(y)\, e^{-ik_0 x_i} & \\ u(-x_i)\, Y_0(y)\, e^{ik_0 x_i} & \\ u(+x_i)\, Y_n(y)\, e^{-k_n x_i} & n = 1, 2, \cdots \\ u(-x_i)\, Y_n(y)\, e^{k_n x_i} & n = 1, 2, \cdots \end{array} \right\} \tag{5.15}$$

which is the vector consisting of the function part of the scattering potential, where $\{\quad\}^T$ means the transpose and $u(x)$ is the unit step function equal to 1 for $x > 0$ and zero for $x < 0$.

By the way, (5.14) and (5.15) are the exact expression, but the effects of evanescent-wave components are very small in practice due to exponential decay except when the bodies are extremely in close proximity. Therefore it is practical to neglect the evanescent components of $n \geq 1$ and to consider only the progressive components in treating the reflected wave by other bodies as an incident wave.

5.2 Diffraction Problem of Multiple Bodies

Let us consider the diffraction problem first, where all bodies are fixed and, as in Fig. 5.1, the incoming wave from outside is assumed to be propagating from the positive x-axis. Normalizing the velocity potential of this incident wave as in (5.1), it can be written as

$$\phi_I(x, y) = Y_0(y)\, e^{ik_0 x} \tag{5.16}$$

Rewriting this with the j-th local coordinate system, we have

$$\begin{aligned} \phi_I(x_j, y) &= e^{ik_0 L_j}\, Y_0(y)\, e^{ik_0 x_j} \\ &= \{0,\, e^{ik_0 L_j}\} \left\{ \begin{array}{l} Y_0(y)\, e^{-ik_0 x_j} \\ Y_0(y)\, e^{ik_0 x_j} \end{array} \right\} \equiv \{a_j\}^T \{\psi_I^j(x_j, y)\} \end{aligned} \tag{5.17}$$

Here $x = x_j + L_j$ (see Fig. 5.1 for the definition of L_j) has been substituted.

As mentioned before, incident waves to the j-th body include not only ϕ_I given above but also reflected waves from other bodies. For instance, the reflected wave by the i-th body, given by (5.14), can be rewritten with the j-th local coordinate system as follows:

$$\left. \begin{array}{l} u(+x_i)\, Y_0(y)\, e^{-ik_0 x_i} = \delta_{ij}\, e^{-ik_0 L_{ij}}\, Y_0(y)\, e^{-ik_0 x_j} \\ u(-x_i)\, Y_0(y)\, e^{ik_0 x_i} = \delta_{ji}\, e^{-ik_0 L_{ij}}\, Y_0(y)\, e^{ik_0 x_j} \end{array} \right\} \tag{5.18}$$

Here we have introduced a special symbol δ_{ij} meaning

$$\delta_{ij} = \left\{ \begin{array}{ll} 1 & i < j \text{ の時} \\ 0 & i > j \text{ の時} \end{array} \right. \tag{5.19}$$

and L_{ij} is, as shown in Fig. 5.1, the distance between the i-th and j-th bodies. Neglecting evanescent-wave components ($n \geq 1$) in the scattering potential of (5.15) by the i-th body and then substituting (5.18) into (5.15), we can obtain the following expression:

$$\begin{aligned} \{\psi_S^i(x_i, y)\} &= \begin{bmatrix} \delta_{ij}\, e^{-ik_0 L_{ij}}, & 0 \\ 0, & \delta_{ji}\, e^{-ik_0 L_{ij}} \end{bmatrix} \begin{Bmatrix} Y_0(y)\, e^{-ik_0 x_j} \\ Y_0(y)\, e^{ik_0 x_j} \end{Bmatrix} \\ &\equiv \big[\, T_{ij} \,\big] \big\{\psi_I^j(x_j, y)\big\} \end{aligned} \tag{5.20}$$

Here $\big\{\psi_I^j(x_j, y)\big\}$ in the above is the vector of "generalized" incident waves, which consists of normalized incident-wave components propagating from the left (for the first term) and from the right (for the second term) of the j-th body, respectively, with unit amplitude. $\big[\, T_{ij} \,\big]$ is called the coordinate transformation matrix.

Considering (5.20) for all bodies except for the j-th body, we can obtain an expression for the total incident wave impinging upon the j-th body in the form

$$\phi_I^j(x_j, y) = \left(\big\{a_j\big\}^T + \sum_{\substack{i=1 \\ i \neq j}}^{N} \big\{A_i\big\}^T \big[\, T_{ij} \,\big] \right) \big\{\psi_I^j(x_j, y)\big\} \tag{5.21}$$

The diffraction characteristics of the j-th body in response to $\big\{\psi_I^j(x_j, y)\big\}$ can be obtained in the same manner as that in the diffraction problem for a single body, simply by substituting the component waves in the "generalized" incident-wave vector as ϕ_I on the right-hand side of (5.11). Specifically, by denoting the scattering potential in response to $\big\{\psi_I^j(x_j, y)\big\}$ as $\big\{\varphi_S^j(x_j, y)\big\}$ and the total diffraction potential as $\big\{\varphi_D^j(x_j, y)\big\}$, the following expression may be obtained

$$\begin{aligned} \big\{\varphi_S^j(x_j, y)\big\} &= \begin{bmatrix} i\, H_4^-(k_0), & i\, H_4^+(k_0) \\ i\, H_4^+(k_0), & i\, H_4^-(k_0) \end{bmatrix} \big\{\psi_S^j(x_j, y)\big\} \\ &\equiv \big[\, B \,\big]^T \big\{\psi_S^j(x_j, y)\big\} \end{aligned} \tag{5.22}$$

where

$$H_4^{\pm}(k_0) = -\, C_0 \int_{S_H} \varphi_D^j(Q) \frac{\partial}{\partial n_Q} Y_0(\eta)\, e^{\pm i k_0 \xi}\, ds \tag{5.23}$$

and $\big[\, B \,\big]$ in (5.22) is referred to as the diffraction characteristics matrix.

Summarizing above, the scattering potential of j-th body in response to the total incident wave given by (5.21) can be expressed in the following form:

$$\phi_S^j(x_j, y) = \left(\big\{a_j\big\}^T + \sum_{\substack{i=1 \\ i \neq j}}^{N} \big\{A_i\big\}^T \big[\, T_{ij} \,\big] \right) \big[\, B \,\big]^T \big\{\psi_S^j(x_j, y)\big\} \tag{5.24}$$

The scattering potential of j-th body is also given by (5.14), with i replaced with j, and that expression of the potential must be the same as (5.24). Thus the following relation can be obtained:

$$\big\{A_j\big\}^T = \big\{a_j\big\}^T \big[\, B \,\big]^T + \sum_{\substack{i=1 \\ i \neq j}}^{N} \big\{A_i\big\}^T \big[\, T_{ij} \,\big] \big[\, B \,\big]^T \tag{5.25}$$

Rewriting this equation by taking the transpose, we can obtain simultaneous equations for the unknown coefficient vector $\big\{A_j\big\}$ of the scattering potential in the form

$$\big\{A_j\big\} - \big[\, B \,\big] \sum_{\substack{i=1 \\ i \neq j}}^{N} \big[\, T_{ij} \,\big]^T \big\{A_i\big\} = \big[\, B \,\big] \big\{a_j\big\} \quad (j = 1 \sim N) \tag{5.26}$$

Now that we could determine the scattering potential, we will consider next the wave-exciting force on each body. The total incident wave impinging upon the j-th body is given by (5.21) and the amplitude part in parentheses is determined by solving (5.26). Thus what we should do is to compute the elementary wave-exciting force to each component in the 'generalized' incident-wave vector $\{\psi_I^j(x_j,y)\}$.

In fact, necessary computations for that purpose have been already finished in the process of computing the matrix $[\,B\,]$, because the diffraction potential $\{\varphi_D^j(x_j,y)\}$ is already computed for computing $\{\varphi_S^j(x_j,y)\}$. Therefore, with this diffraction potential to the 'generalized' incident-wave vector, the elementary wave-exciting-force vector acting in the k-th direction can be computed from

$$\{E_k^j\} = \int_{S_H} \{\varphi_D^j(x_j,y)\}\, n_k\, ds \tag{5.27}$$

Multiplying this by the amplitude of total incident wave, the wave-exciting force in the k-th direction on the j-th body can be computed in the form

$$W_k^j = \rho g a \left(\{a_j\}^T + \sum_{\substack{i=1 \\ i\neq j}}^{N} \{A_i\}^T \,[\,T_{ij}\,] \right) \{E_k^j\} \tag{5.28}$$

5.3 Diffraction Problem of a Catamaran

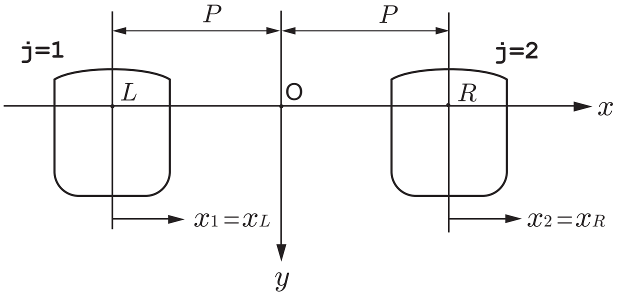


Fig. 5.2 Notations in catamaran problem.

The theory in the preceding section can be applied irrespective of the number of floating bodies, but for clear understanding let us consider the simplest but practical case of $N = 2$; that is, the case of catamaran. For brevity the water depth is assumed infinite; that is, $k_0 = K$ and $C_0 = 1$.

The separation distance between the centerlines of demihull is, as shown in Fig. 5.2, denoted as $L_{12} = 2P$, and L_j used in (5.17) is given by $L_1 = -P$ and $L_2 = P$. When necessary, notations x_L and x_R will be used instead of x_1 and x_2, respectively.

With these notations, (5.17), (5.20), and (5.22) can be written for a catamaran as follows:

$$\{a_1\} = \left\{ \begin{array}{c} 0 \\ e^{-iKP} \end{array} \right\}, \quad \{a_2\} = \left\{ \begin{array}{c} 0 \\ e^{iKP} \end{array} \right\}, \quad [\,B\,] = \left[\begin{array}{cc} i\,H_4^-, & i\,H_4^+ \\ i\,H_4^+, & i\,H_4^- \end{array} \right] \tag{5.29}$$

$$[\,T_{12}\,]^T = \left[\begin{array}{cc} e^{-i2KP}, & 0 \\ 0 & 0 \end{array} \right], \quad [\,T_{21}\,]^T = \left[\begin{array}{cc} 0 & 0 \\ 0 & e^{-i2KP} \end{array} \right] \tag{5.30}$$

Therefore, (5.26) can be written specifically in a form of simultaneous equations for four unknowns.

$$A_1^+ - i\,H_4^+\, e^{-i2KP}\, A_2^- = i\,H_4^+\, e^{-iKP} \tag{5.31}$$

$$A_1^- - i\,H_4^-\, e^{-i2KP}\, A_2^- = i\,H_4^-\, e^{-iKP} \tag{5.32}$$

$$A_2^+ - i\,H_4^-\, e^{-i2KP}\, A_1^+ = i\,H_4^+\, e^{iKP} \tag{5.33}$$

$$A_2^- - i\,H_4^+\, e^{-i2KP}\, A_1^+ = i\,H_4^-\, e^{iKP} \tag{5.34}$$

From (5.31) and (5.34), we can solve for A_1^+ and A_2^- and the results can be expressed as

$$A_1^+ = \frac{i\,H_4^+\big(1 + i\,H_4^-\big)\,e^{-iKP}}{1 + \big\{H_4^+ e^{-i2KP}\big\}^2} \tag{5.35}$$

$$A_2^- = -\,e^{iKP} + \frac{\big(1 + i\,H_4^-\big)\,e^{iKP}}{1 + \big\{H_4^+ e^{-i2KP}\big\}^2} \tag{5.36}$$

Here A_1^+ is the complex amplitude of the wave generated by body L and propagating to body R. From (5.18), this quantity can be expressed with the local coordinate system at body R in the form

$$A_1^+\,e^{-i2KP} \equiv D_R(K) = \frac{i\,H_4^+\,\big(1 + i\,H_4^-\big)\,e^{-i3KP}}{1 + \big\{H_4^+ e^{-i2KP}\big\}^2} \tag{5.37}$$

On the other hand, A_2^- is the complex amplitude of the wave generated by body R and propagating to body L. Combining this wave with the incident wave (which is assumed to propagate in the same negative x-axis), we can write from (5.17) and(5.18) the total wave with the local coordinate system at body L as follows:

$$e^{-iKP} + A_2^-\,e^{-i2KP} \equiv D_L(K) = \frac{\big(1 + i\,H_4^-\big)\,e^{-iKP}}{1 + \big\{H_4^+ e^{-i2KP}\big\}^2} \tag{5.38}$$

This result is the same as that derived originally by Ohkusu with the concept of infinite number of wave reflections between two demihulls.

Having obtained A_1^+ and A_2^-, we may substitute these in (5.32) and (5.33) and then obtain the results of A_1^- and A_2^+ in the form

$$A_1^- = i\,H_4^-\,e^{-iKP} + i\,H_4^-\,A_2^-\,e^{-i2KP} = i\,H_4^-\,D_L(K) \tag{5.39}$$

$$A_2^+ = i\,H_4^+\,e^{iKP} + i\,H_4^-\,A_1^+\,e^{-i2KP} = i\,H_4^+\,e^{iKP} + i\,H_4^-\,D_R(K) \tag{5.40}$$

Up to this point, the problem has been solved completely. Next, in terms of the results so far, let us obtain the reflection wave (R) and transmission wave (T) coefficients. By recalling that the reflection wave coefficient is the wave amplitude at $x \to +\infty$, it takes the following form with the global coordinate system O-xy

$$\begin{aligned} R &= A_1^+\,e^{-iKP} + A_2^+\,e^{iKP} \\ &= i\,H_4^+\,e^{i2KP} + \big(1 + i\,H_4^-\big)\,D_R(K)\,e^{iKP} \end{aligned} \tag{5.41}$$

On the other hand, the transmission wave coefficient is defined at $x \to -\infty$ including the incident wave, and thus it takes the following form

$$\begin{aligned} T &= 1 + A_1^-\,e^{iKP} + A_2^-\,e^{-iKP} \\ &= \big(1 + i\,H_4^-\big)\,D_L(K)\,e^{iKP} \end{aligned} \tag{5.42}$$

Next let us consider the wave-exciting force. As the first step, we consider $\big\{E_k^j\big\}$ defined by (5.27). This is the elementary wave-exciting-force vector in response to each component in $\big\{\psi_I^j(x_j, y)\big\}$, and as one can understand from (5.20), each component in $\big\{\psi_I^j(x_j, y)\big\}$ corresponds to the incident wave incoming from the negative and positive x-axis, respectively. Therefore, according to Haskind-Newman's relation, the following result must be readily obtained:

$$\{E_k^j\} = \{H_k^-(K),\ H_k^+(K)\} \quad (k = 1 \sim 3) \tag{5.43}$$

Here $H_k^{\pm}(K)$ denotes the Kochin function in the radiation problem of the k-th mode of motion. Since $k = 1, 2, 3$ correspond to sway, heave, and roll respectively, $H_k^-(K) = (-1)^k H_k^+(K)$ holds for a symmetric floating body.

Using this relation together with (5.29) and (5.30), we can write (5.28) explicitly as follows:

$$W_k^L = \rho g a \left\{ e^{-iKP} + A_2^- \, e^{-i2KP} \right\} H_k^+(K) \\ = \rho g a \, D_L(K) \, H_k^+(K) \tag{5.44}$$

$$W_k^R = \rho g a \left\{ e^{iKP} H_k^+(K) + A_1^+ \, e^{-i2KP} H_k^-(K) \right\} \\ = \rho g a \left\{ e^{iKP} + (-1)^k D_R(K) \right\} H_k^+(K) \tag{5.45}$$

5.4 Radiation Problem of Multiple Bodies

The basic idea for considering multiple-body hydrodynamic interactions in the radiation problem can be the same as that in the diffraction problem. Only the difference is to view the wave radiated by oscillation of another body as the incident wave in place of the incoming wave from outside.

Let us consider the radiation wave by forced oscillation in the ℓ-th mode of motion of the n-th body, with its velocity potential expressed as

$$\Phi(x, y, t) = \mathrm{Re}\Big[i\omega X_\ell^n \, \varphi_\ell^n(x_n, y) \, e^{i\omega t} \Big] \\ = \mathrm{Re}\Big[\frac{g}{i\omega} (-K X_\ell^n) \, \varphi_\ell^n(x_n, y) \, e^{i\omega t} \Big] \tag{5.46}$$

where X_ℓ^n denotes the complex motion amplitude. We note that the summation sign with respect to mode index ℓ is omitted for simplicity with summation convention.

The solution of $\varphi_\ell^n(x_n, y)$ can be obtained in a form of (5.12), but as we did before, local waves may be neglected except near the n-th body. Then the solution can be expressed in the form

$$\varphi_\ell^n(x_n, y) = \left\{ i \, H_\ell^+(k_0), \; i \, H_\ell^-(k_0) \right\} \left\{ \psi_S^n(x_n, y) \right\} \\ \equiv \left\{ b_\ell \right\}^T \left\{ \psi_S^n(x_n, y) \right\} \tag{5.47}$$

where
$$H_\ell^\pm(k_0) = C_0 \int_{S_H} \left\{ \frac{\partial \varphi_\ell(Q)}{\partial n_Q} - \varphi_\ell(Q) \frac{\partial}{\partial n_Q} \right\} Y_0(\eta) \, e^{\pm i k_0 \xi} \, ds \tag{5.48}$$

is the Kochin function for the ℓ-th mode of the radiation problem; which must be the same irrespective of the body number n, if the body geometry of all bodies is the same. With this reason, its vector is denoted simply as $\left\{ b_\ell \right\}$ in (5.47).

By applying (5.20), the velocity potential (5.47) can be written with the local coordinate system at the j-th body as follows:

$$\varphi_\ell^j(x_j, y) = \left\{ b_\ell \right\}^T \left[T_{nj} \right] \left\{ \psi_I^j(x_j, y) \right\} \tag{5.49}$$

Here we should note that this 'incident wave' is radiated from the n-th body and thus $\left[T_{nn} \right] = 0$ for the case of $j = n$.

Since the quantity of (5.49) multiplied by $-K X_\ell^n$ can be regarded as an incident wave corresponding to (5.17), the scattered velocity potential at j-th body including multiple-body interactions can be expressed, as in (5.24), in the form

$$\phi_S^j(x_j, y) = -K X_\ell^n \left(\left\{ b_\ell \right\}^T \left[T_{nj} \right] + \sum_{\substack{i=1 \\ i \neq j}}^{N} \left\{ A_i \right\}^T \left[T_{ij} \right] \right) \left[B \right]^T \left\{ \psi_S^j(x_j, y) \right\} \tag{5.50}$$

Similarly, the simultaneous equations for unknown coefficient vector $\{A_j\}$ can be given in a similar form to (5.26) and its result is expressed as

$$\{A_j\} - [B] \sum_{\substack{i=1 \\ i \neq j}}^{N} [T_{ij}]^T \{A_i\} = [B] [T_{nj}]^T \{b_\ell\} \qquad (j = 1 \sim N) \tag{5.51}$$

The hydrodynamic force associated with multiple-body interactions can be given in a similar form to (5.28), with amplitude a replaced with $-KX_\ell^n$. Thus it takes the form

$$F_{k\ell}^{jn} = -\rho g\, K X_\ell^n \left(\{b_\ell\}^T [T_{nj}] + \sum_{\substack{i=1 \\ i \neq j}}^{N} \{A_i\}^T [T_{ij}] \right) \{E_k^j\} \tag{5.52}$$

This hydrodynamic force should be interpreted as the interaction force on the j-th body in the k-th direction when the n-th body oscillates in the ℓ-th mode. In the radiation problem, in addition to the interaction force derived above, we must add the hydrodynamic radiation force (added mass and damping coefficient) on the n-th body oscillating as a single body.

5.5 Radiation Problem of a Catamaran

Let us consider specifically the case of catamaran. For simplicity, the water depth is assumed to be infinite. First, we consider the case of $n = 1$, i.e. when body L oscillates. Since the left-hand side of (5.51) is completely the same as that for the diffraction problem, we can obtain the following equations corresponding to (5.31)∼(5.34):

$$A_1^+ - i\, H_4^+\, e^{-i2KP} A_2^- = 0 \tag{5.53}$$

$$A_1^- - i\, H_4^+\, e^{-i2KP} A_2^- = 0 \tag{5.54}$$

$$A_2^+ - i\, H_4^+\, e^{-i2KP} A_1^+ = i\, H_4^- \left(i\, H_\ell^+ e^{-i2KP} \right) \tag{5.55}$$

$$A_2^- - i\, H_4^+\, e^{-i2KP} A_1^+ = i\, H_4^+ \left(i\, H_\ell^+ e^{-i2KP} \right) \tag{5.56}$$

Solving these equations, we can obtain solutions for the unknown coefficients in the scattered wave and the results are expressed as

$$\left(i\, H_\ell^+ + A_1^+ \right) e^{-i2KP} = i\, H_\ell^+ \frac{e^{-i2KP}}{1 + \left\{ H_4^+ e^{-i2KP} \right\}^2} \equiv i\, H_\ell^+\, E_R(K) \tag{5.57}$$

$$A_2^-\, e^{-i2KP} = i\, H_\ell^+ \frac{i\, H_4^+ e^{-i4KP}}{1 + \left\{ H_4^+ e^{-i2KP} \right\}^2} \equiv i\, H_\ell^+\, E_L(K) \tag{5.58}$$

$$A_1^- = i\, H_4^-\, e^{-i2KP} A_2^- = i\, H_\ell^+ \left\{ i\, H_4^-\, E_L(K) \right\} \tag{5.59}$$

$$A_2^+ = i\, H_4^-\, e^{-i2KP} \left(i\, H_\ell^+ + A_1^+ \right) = i\, H_\ell^+ \left\{ i\, H_4^-\, E_R(K) \right\} \tag{5.60}$$

With these results, the interaction force can be computed from (5.52). Specifically the hydrodynamic interaction forces on body $j = 1$ (body L) and body $j = 2$ (body R) due to forced oscillation of body L in the ℓ-th mode can be obtained as follows:

$$\begin{aligned} F_{k\ell}^{LL} &= -\rho g\, K X_\ell^L \left(A_2^- e^{-i2KP} \right) H_k^+ \\ &= -\rho g\, K X_\ell^L \left\{ iE_L(K) \right\} H_k^+ H_\ell^+ \equiv X_\ell^L f_{k\ell} \end{aligned} \tag{5.61}$$

$$\begin{aligned} F_{k\ell}^{RL} &= -\rho g\, K X_\ell^L \left(i\, H_\ell^+ + A_1^+ \right) e^{-i2KP} H_k^- \\ &= -\rho g\, K X_\ell^L (-1)^k \left\{ iE_R(K) \right\} H_k^+ H_\ell^+ \equiv X_\ell^L (-1)^k g_{k\ell} \end{aligned} \tag{5.62}$$

Here
$$
\left.\begin{aligned}
f_{k\ell} &= -\rho g K \{ iE_L(K) \} H_k^+ H_\ell^+ = f_{\ell k} \\
g_{k\ell} &= -\rho g K \{ iE_R(K) \} H_k^+ H_\ell^+ = g_{\ell k}
\end{aligned}\right\} \tag{5.63}
$$

and it is worthwhile to note that the symmetry relation holds.

Similarly the progressive wave at infinity can be computed. Writing with the global coordinate system O-xy, the progressive waves at $x \to \pm\infty$ can be obtained and expressed as follows:

$$
\begin{aligned}
\zeta_{+\infty}^L &= -KX_\ell^L \left[\left(i\,H_\ell^+ + A_1^+ \right) e^{-iKP} + A_2^+ \, e^{iKP} \right] \\
&= -KX_\ell^L \left\{ \left(1 + i\,H_4^-\right) E_R(K) \right\} i\,H_\ell^+ \, e^{iKP} \equiv -KX_\ell^L \, \alpha_\ell
\end{aligned} \tag{5.64}
$$

$$
\begin{aligned}
\zeta_{-\infty}^L &= -KX_\ell^L \left[\left(i\,H_\ell^- + A_1^- \right) e^{iKP} + A_2^- e^{-iKP} \right] \\
&= -KX_\ell^L \left\{ (-1)^\ell + \left(1 + i\,H_4^-\right) E_L(K) \right\} i\,H_\ell^+ \, e^{iKP} \equiv -KX_\ell^L \, \beta_\ell
\end{aligned} \tag{5.65}
$$

where
$$
\left.\begin{aligned}
\alpha_\ell &= \left(1 + i\,H_4^-\right) E_R(K) \, i\,H_\ell^+ \, e^{iKP} \\
\beta_\ell &= \left\{ (-1)^\ell + \left(1 + i\,H_4^-\right) E_L(K) \right\} i\,H_\ell^+ \, e^{iKP}
\end{aligned}\right\} \tag{5.66}
$$

In the same manner, let us consider the case of $n = 2$, i.e. when body R oscillates in the ℓ-th mode. The calculation procedure for this case is almost the same as the previous case and the solutions can be obtained in the following form

$$
A_1^+ \, e^{-i2KP} = i\,H_\ell^- \, E_L(K) \tag{5.67}
$$

$$
\left(i\,H_\ell^- + A_2^- \right) e^{-i2KP} = i\,H_\ell^- \, E_R(K) \tag{5.68}
$$

$$
A_1^- = i\,H_\ell^- \left\{ i\,H_4^- E_R(K) \right\} \tag{5.69}
$$

$$
A_2^+ = i\,H_\ell^- \left\{ i\,H_4^- E_L(K) \right\} \tag{5.70}
$$

The hydrodynamic interaction forces can be computed by substituting these results in (5.52) and expressed as

$$
F_{k\ell}^{LR} = -\rho g\, KX_\ell^R (-1)^\ell \{ iE_R(K) \} H_k^+ H_\ell^+ = X_\ell^R (-1)^\ell \, g_{k\ell} \tag{5.71}
$$

$$
F_{k\ell}^{RR} = -\rho g\, KX_\ell^R (-1)^{k+\ell} \{ iE_L(K) \} H_k^+ H_k^- = X_\ell^R (-1)^{k+\ell} \, f_{k\ell} \tag{5.72}
$$

The progressive waves at $x \to \pm\infty$ can be computed in the same way and expressed as

$$
\begin{aligned}
\zeta_{+\infty}^R &= -KX_\ell^R \left\{ (-1)^\ell + \left(1 + i\,H_4^-\right) E_L(K) \right\} (-1)^\ell \, i\,H_\ell^+ \, e^{iKP} \\
&= -KX_\ell^R (-1)^\ell \, \beta_\ell
\end{aligned} \tag{5.73}
$$

$$
\begin{aligned}
\zeta_{-\infty}^R &= -KX_\ell^R \left\{ \left(1 + i\,H_4^-\right) E_R(K) \right\} (-1)^\ell \, i\,H_\ell^+ \, e^{iKP} \\
&= -KX_\ell^R (-1)^\ell \, \alpha_\ell
\end{aligned} \tag{5.74}
$$

From the results obtained above, the total interaction force on body L (denoted as $\mathcal{L}_{k\ell}$) must be sum of (5.61) and (5.71) and the total interaction force on body R (denoted as $\mathcal{R}_{k\ell}$) must be sum of (5.62) and (5.72). Their results can be written in the form

$$
\mathcal{L}_{k\ell} = F_{k\ell}^{LL} + F_{k\ell}^{LR} = X_\ell^L \, f_{k\ell} + X_\ell^R (-1)^\ell \, g_{k\ell} \tag{5.75}
$$

$$
\mathcal{R}_{k\ell} = F_{k\ell}^{RL} + F_{k\ell}^{RR} = \left[X_\ell^L \, g_{k\ell} + X_\ell^R (-1)^\ell \, f_{k\ell} \right] (-1)^k \tag{5.76}
$$

In the same way, the total progressive wave at $x \to \pm\infty$ must be sum of (5.64) and (5.73) at $x \to +\infty$ and (5.65) and (5.74) at $x \to -\infty$. Those results are given as follows:

$$\zeta_{+\infty} = \zeta^L_{+\infty} + \zeta^R_{+\infty} = -K \left[X^L_\ell \, \alpha_\ell + X^R_\ell (-1)^\ell \, \beta_\ell \right] \tag{5.77}$$

$$\zeta_{-\infty} = \zeta^L_{-\infty} + \zeta^R_{-\infty} = -K \left[X^L_\ell \, \beta_\ell + X^R_\ell (-1)^\ell \, \alpha_\ell \right] \tag{5.78}$$

6. Numerical Computations Based on Free-Surface Green Function Method

In order to compute hydrodynamic forces acting on a body and the amplitude function (Kochin function) of body-generated waves, the velocity potential on the body surface (which is equivalent to the pressure) must be obtained. In this chapter, explanation will be made on a numerical calculation method based on the boundary element (or free-surface Green function) method.

6.1 Boundary Integral Equation

The boundary integral equation to be solved can be derived from Gauss' theorem and its result is known as Green's formula. As already explained in Chapter 2, we introduce first the **free-surface Green function** which satisifes the same homogeneous bounday conditions as those to be satisfied by the velocity potential. In that sense, the calculation method which will be explained in this chapter is referred to as the boundary integral equation method or free-surface Green function method.

The derivation for the integral equation to be solved has already been described, and the resultant integral equations to be solved are (2.49) for the radiation problem and (3.21) for the diffraction problem. Those can be written in a unified form as follows:

$$\frac{1}{2}\,\varphi_j(\mathrm{P}) + \int_{S_H} \varphi_j(\mathrm{Q}) \frac{\partial}{\partial n_{\mathrm{Q}}}\, G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q}) = \left\{ \begin{array}{l} \displaystyle\int_{S_H} n_j(\mathrm{Q})\, G(\mathrm{P};\mathrm{Q})\, ds(\mathrm{Q})\,, \quad j = 1 \sim 3 \\ \varphi_0(\mathrm{P})\,, \quad j = D \end{array} \right. \tag{6.1}$$

where $\mathrm{P} = (x, y),\ \mathrm{Q} = (\xi, \eta)$ and $G(\mathrm{P};\mathrm{Q})$ denotes the free-surface Green function. Since the Green function satisifies all homogeneous boundary conditions, the integral surface in (6.1) is only the wetted surface of a body (S_H) on which the bounday condition is inhomogeneous, as given specifically by (3.9).

6.2 Free-Surface Green Function

There are several ways for deriving the free-surface Green function $G(\mathrm{P};\mathrm{Q})$, but probably the simplest among them may be the use of Fourier transform; its detail was already described in Section 2.1. Here as the representative from various expressions, let us rewrite (2.33):

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \log \frac{r}{r_1} - \frac{1}{\pi} \int_0^\infty \frac{k \cos k(y+\eta) - K \sin k(y+\eta)}{k^2 + K^2}\, e^{-k|x-\xi|}\, dk \\ + i\, e^{-K(y+\eta) - iK|x-\xi|} \tag{6.2}$$

where

$$\left.\begin{array}{c} r \\ r_1 \end{array}\right\} = \sqrt{(x-\xi)^2 + (y \mp \eta)^2} \tag{6.3}$$

The term $\frac{1}{2\pi}\log r$ in the above is known as the fundamental (or principal) solution of the 2D Laplace equation; that is, the velocity potential due to a hydrodynamic source with unit strength, and the remaining terms are supplemented to satisfy the free-surface and radiation conditions. It should be noted that the first line in (6.2) represents the **local wave** (or evanescent wave) which decays as $|x - \xi| \to \infty$, and the second line (last term) in (6.2) represents the **progressive wave** propagating outwards from the source point.

Now we must consider how to evaluate numerically (6.2) particularly the local-wave integral term with respect to k. In order to explain necessary mathematical transformation, let us consider the following integral denoted as $\mathcal{I}$

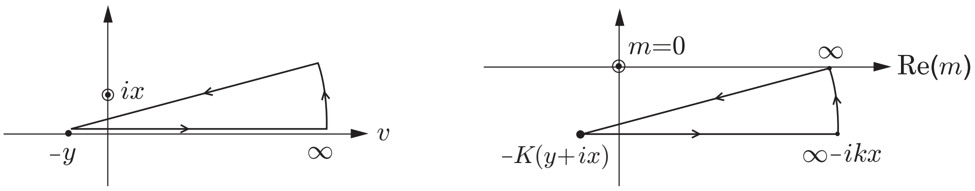
$$\mathcal{I} \equiv \int_0^\infty \frac{k\cos ky - K\sin ky}{k^2+K^2}\, e^{-k|x|}\, dk \tag{6.4}$$

This function satisfies the following differential equation:

$$\frac{d\mathcal{I}}{dy} + K\,\mathcal{I} = -\int_0^\infty e^{-k|x|}\sin ky\, dk = \frac{-y}{x^2+y^2} \tag{6.5}$$

Therefore its solution can be readily obtained and written as follows:

$$\mathcal{I} = e^{-Ky}\left[\int_{-\infty}^{y} \frac{-\eta}{x^2+\eta^2}\, e^{K\eta}\, d\eta\right] = e^{-Ky}\int_{-y}^{\infty}\frac{v}{x^2+v^2}\, e^{-Kv}\, dv = e^{-Ky}\,\mathrm{Re}\int_{-y}^{\infty}\frac{e^{-Kv}}{v-ix}\, dv \tag{6.6}$$



Integration path in the complex plane for $x>0$

Fig.6.1 Deformation of integration path in the complex plane.

With variable transformation of $K(v-ix)=m$, the integration path in the complex m-plane may be taken as shown in Fig. 6.1. Since there is no singularity inside the enclosed integration path, the residue theorem provides the following result:

$$\mathcal{I} = \mathrm{Re}\left[e^{-K(y+ix)}\int_{-K(y+ix)}^{\infty}\frac{e^{-m}}{m}\, dm\right] = \mathrm{Re}\Big[e^{-z}E_1(-z)\Big] \tag{6.7}$$

where

$$E_1(\zeta) = \int_\zeta^\infty \frac{e^{-t}}{t}\, dt\,, \quad z = K(y+ix) \tag{6.8}$$

Here $E_1(-z)$ denotes the exponential integral function with complex variable; its computation method is well studied and summarized in Appendix A2, with which we can perform the fast computation with desired accuracy.

To summarize, the free-surface Green function can be expressed as follows:

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\left\{\log\frac{r}{r_1} - 2\,F_C(x-\xi, y+\eta)\right\} \tag{6.9}$$

where

$$\left.\begin{aligned} F_C(x-\xi,y+\eta) &= \mathrm{Re}\Big[e^{-Z}E_1(-Z)\Big] - i\pi\, e^{-Z} \\ Z &= K(y+\eta) + iK\big|x-\xi\big| \end{aligned}\right\} \tag{6.10}$$

In solving the integral equation, as will be shown in the next section, it may be expedient to introduce a function which is conjugate to $F_C(x-\xi,y+\eta)$ defined by (6.10). This conjugate function $F_S(x-\xi,y+\eta)$ must satisfy the following relations:

$$\frac{\partial F_C}{\partial n} = \frac{\partial F_S}{\partial s}\,, \quad \frac{\partial F_C}{\partial \xi} = \frac{\partial F_S}{\partial \eta}\,, \quad \frac{\partial F_C}{\partial \eta} = -\frac{\partial F_S}{\partial \xi} \tag{6.11}$$

Thus it may be easier to confirm that the desired function takes the following form

$$F_S(x-\xi, y+\eta) = \mathrm{sgn}(x-\xi)\Big\{ -\int_0^\infty \frac{k\sin k(y+\eta) + K\cos k(y+\eta)}{k^2+K^2}\, e^{-k|x-\xi|}\, dk \\ - \pi\, e^{-K(y+\eta)-iK|x-\xi|} \Big\}$$

$$= \mathrm{sgn}(x-\xi)\Big\{ \mathrm{Im}\Big[e^{-Z} E_1(-Z) \Big] - \pi\, e^{-Z} \Big\} \tag{6.12}$$

It may be noteworthy that r_1 appearing in the first term on the right-hand side of (6.9) is given by reversing the sign of η in r, but in fact it can also be given by reversing the sign of y, which will be convenient when integrating with respect to $\mathrm{Q} = (\xi, \eta)$.

6.3 Numerical Solution Method for Integral Equation

In fact there exist several different methods for solving the boundary integral equation (6.1); among them a fundamental and commonly used method, i.e. the constant panel method using zeroth order element combined with the collocation method will be explained in this section.

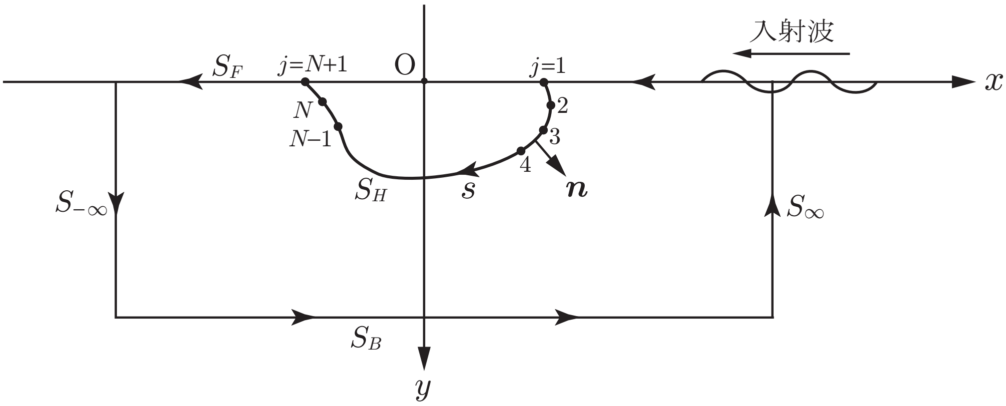


Fig.6.2 Coordinate system and notations for numerical computations.

First, as shown in Fig. 6.2, the body surface in $y > 0$ is divided into N elements i.e. segments (s_n; $n = 1 \sim N$), and we assume that the velocity potential on each segment is of constant value. In this case, the number of unknowns is N. Thus by selecting N different points of $\mathrm{P}(x, y)$ at which both sides of (6.1) are enforced to be equal, the integral equation (6.1) may be converted into a linear system of simultaneous equations with $N \times N$ matrix and then solved using a conventional method for simultaneous equations. This solution method is referred to as the **collocation method**, in which the positions as N different points of $\mathrm{P}(x, y)$ are normally placed at the center of each segment.

Multiplying both sides of (6.1) by 2π and then adopting the zeroth-order element and collocation method, we can recast the integral equation in the following discretized form:

$$\pi\,\varphi_j(\mathrm{P}_m) + \sum_{n=1}^{N} \varphi_j(\mathrm{Q}_n)\, D_{mn} = \left\{ \begin{array}{ll} \displaystyle\sum_{n=1}^{N} n_j(\mathrm{Q}_n)\, S_{mn} & (j = 1 \sim 3) \\ 2\pi\,\varphi_0(\mathrm{P}_m) & (j = D) \end{array} \right. \tag{6.13}$$

Here $m = 1 \sim N$ and the matrix coefficients in the above are defined as follows:

$$D_{mn} = \int_{s_n} \frac{\partial}{\partial n_{\mathrm{Q}}} \Big\{ \log\frac{r}{r_1} - 2F_C(x_m-\xi, y_m+\eta) \Big\}\, ds(\xi, \eta)\,, \tag{6.14}$$

$$S_{mn} = \int_{s_n} \left\{ \log \frac{r}{r_1} - 2F_C(x_m - \xi, y_m + \eta) \right\} ds(\xi, \eta) \tag{6.15}$$

Let us describe how to evaluate analytically these matrix coefficients on each segment. As shown in Fig. 6.3, we introduce a local polar coordinate system (r, δ) with the origin placed at a nodal point (ξ_n, η_n). Then, since the value of δ is constant on the segment, the integrals of D_{mn} and S_{mn} on the segment will be functions of r only. We can write (ξ, η) as

$$\left.\begin{aligned} \xi &= \xi_n + r\cos\delta \\ \eta &= \eta_n + r\sin\delta \end{aligned}\right\} \tag{6.16}$$

with $\cos\delta$ and $\sin\delta$ given by

$$\left.\begin{aligned} \cos\delta &= \frac{\xi_{n+1} - \xi_n}{D}, \ \sin\delta = \frac{\eta_{n+1} - \eta_n}{D} \\ D &= \sqrt{(\xi_{n+1} - \xi_n)^2 + (\eta_{n+1} - \eta_n)^2} \end{aligned}\right\}. \tag{6.17}$$

Then we can see that the range of integration with respect to r is from $r = 0$ to $r = D$.

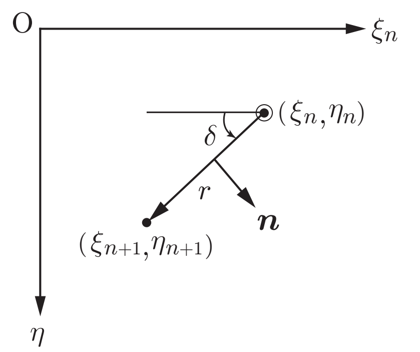


Fig. 6.3 Definition of local coordinate system.

In terms of (6.17), the components of normal vector on a segment can be computed as follows:

$$\left.\begin{aligned} n_x &= n_1 = \sin\delta \\ n_y &= n_2 = -\cos\delta \end{aligned}\right\} \tag{6.18}$$

Details of analytical integration are shown in the appendices in the reference book written in a footnote; Appendix A 2.1 for the integrals related to the logarithmic function $\log r - \log r_1$ and Appendix A 2.2 for the integrals related to the free-surface-effect function $F_C(x_m - \xi, y_m + \eta)$. For brevity in expressions, (x_m, y_m) will be written simply as (x, y) by omitting subscript m. Then the results after analytical integration over a segment for the matrix coefficients D_{mn} and S_{mn} can be summarized as follows:

$$D_{mn} = \bigl(1 - \delta_{mn}\bigr)\mathcal{T}_n(x, y) - \mathcal{T}_n(x, -y) - 2\Bigl[F_S(x - \xi, y + \eta) \Bigr]_n^{n+1} \tag{6.19}$$

$$S_{mn} = \mathcal{L}_n(x, y) - \mathcal{L}_n(x, -y) - 2\,\widehat{S}_n(x, y) \tag{6.20}$$

where

$$\mathcal{T}_n(x, y) = \frac{(x - \xi_n)\sin\delta - (y - \eta_n)\cos\delta}{|(x - \xi_n)\sin\delta - (y - \eta_n)\cos\delta|} \left[\tan^{-1} \frac{(x - \xi)\cos\delta + (y - \eta)\sin\delta}{|(x - \xi_n)\sin\delta - (y - \eta_n)\cos\delta|} \right]_n^{n+1} \tag{6.21}$$

$$\begin{aligned} \mathcal{L}_n(x, y) = &- \left[\Bigl\{ (x - \xi)\cos\delta + (y - \eta)\sin\delta \Bigr\} \log\sqrt{(x - \xi)^2 + (y - \eta)^2} \right]_n^{n+1} \\ &- \Bigl| (x - \xi_n)\sin\delta - (y - \eta_n)\cos\delta \Bigr| \left[\tan^{-1} \frac{(x - \xi)\cos\delta + (y - \eta)\sin\delta}{|(x - \xi_n)\sin\delta - (y - \eta_n)\cos\delta|} \right]_n^{n+1} \end{aligned} \tag{6.22}$$

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成山堂書店，ISBN4-425-71321-4

$$\widehat{S}_n(x,y) = -\frac{1}{K}\left[\sin\delta\,\log\sqrt{(x-\xi)^2+(y+\eta)^2} + \cos\delta\,\tan^{-1}\frac{y+\eta}{x-\xi}\right]_n^{n+1}$$
$$-\frac{1}{K}\left[\sin\delta\,F_C(x-\xi,y+\eta) - \cos\delta\,F_S(x-\xi,y+\eta)\right]_n^{n+1} \tag{6.23}$$

In the above, $\left[\,\cdots\,\right]_n^{n+1}$ means the difference of the quantity in brackets evaluated at $\mathrm{Q}_{n+1} = (\xi_{n+1},\eta_{n+1})$ and $\mathrm{Q}_n = (\xi_n,\eta_n)$ must be taken. δ_{mn} in (6.19) denotes Kroenecker's delta, equal to 1 for $m = n$ and zero otherwise. Thus the first term on the right-hand side of (6.19) becomes zero for $m = n$. $F_S(x-\xi,y+\eta)$ appearing in (6.19) and (6.23) is the conjugate function to $F_C(x-\xi,y+\eta)$; its calculation formula is already described in (6.12). We can see from above results that the matrix coefficients in (6.13) can be computed only in terms of logarithmic, arctangent, and exponential integral functions.

By the way, there will be an unfortunate case where the matrix determinant becomes almost zero at a certain special frequency and the solution is indeterminate. This special frequency is referred to as **irregular frequency**, around which resultant hydrodynamic forces computed with obtained velocity potential become unreasonable. Several methods have been proposed to get rid of irregular frequencies. In this section, a slightly modified version of Haraguchi-Ohmatsu's method will be explained briefly and incorporated in the computer code to be explained afterward.

First we should recognize that the occurrence of irregular frequency is not physical but mathematical due to the characteristic of a matrix and thus cannot be removed simply by increasing the number of divided panels. Therefore, to resolve this problem, it is necessary to change the matrix characteristic by adding an integral equation which is similar but different in nature. For that purpose, we put the field point P on the interior free surface inside the floating body (which is outside of the fluid region considered). In this case, the first term on the left-hand side of (6.13) must be zero, which makes the matrix characteristic change so that a stable solution can be obtained. With this idea, let us add a few extra equations with the field point P_m taken on the interior free surface and hence the first term $\pi\varphi_j(\mathrm{P}_m)$ being zero. Denoting the right-hand side of (6.13) as $\mathcal{R}_{jm}$ and $\varphi_j(\mathrm{Q}_n)$ on the left-hand side simply as φ_j^n, we can obtain the following expression for overconstrained simultaneous equations

$$\sum_{n=1}^{N}\mathcal{D}_{mn}\,\varphi_j^n = \mathcal{R}_{jm} \quad (m = 1\sim N, N+1,\cdots,M) \tag{6.24}$$

where

$$\mathcal{D}_{mn} = \begin{cases} \pi\delta_{mn} + D_{mn} & (m = 1\sim N) \\ D_{mn} & (m = N+1\sim M) \end{cases} \tag{6.25}$$

and the field points taken on the interior free surface are numbered as $m = N+1,\cdots,M$. (In reality, taking $3\sim 5$ additional points on the interior free surface would be sufficient.)

Since the number (M) of equations is larger than the number (N) of unknowns in (6.24), we must use the least-squares method to solve (6.24). To apply that method, we consider a square of the error (difference) defined by

$$E \equiv \sum_{m=1}^{M}\left[\sum_{n=1}^{N}\mathcal{D}_{mn}\,\varphi_j^n - \mathcal{R}_{jm}\right]^2, \quad N < M \tag{6.26}$$

Then, applying the conditions $\partial E/\partial\varphi_j^k = 0$ $(k = 1,2,\cdots,N)$ for minimizing squared error, we can obtain a modified system of simultaneous equations in the following form:

$$\sum_{n=1}^{N}\left\{\sum_{m=1}^{M}\mathcal{D}_{mn}\,\mathcal{D}_{mk}\right\}\varphi_j^n = \sum_{m=1}^{M}\mathcal{R}_{jm}\,\mathcal{D}_{mk} \quad \text{for } k = 1\sim N \tag{6.27}$$

We can see that this equation provides N simultaneous equations for N unknowns, and thus we can solve this by using a conventional method like Gauss' elimination method and determine the velocity potential on the body surface.

6.4 Hydrodynamic Forces and Kochin Function

Once the velocity potential on the body surface has been obtained, we can compute hydrodynamic forces and the Kochin function. Since the velocity potential is assumed constant on each segment of the body surface, the added mass (A_{ij}) and damping coefficient (B_{ij}) in nondimensional form may be computed from the following:

$$\begin{aligned} Z'_{ij} \equiv A'_{ij} - i\, B'_{ij} &= \frac{A_{ij}}{\rho b^2 \epsilon_i \epsilon_j} - i\, \frac{B_{ij}}{\rho \omega b^2 \epsilon_i \epsilon_j} \\ &= -\int_{S_H} \varphi_j(x,y) n_i\, ds = -\sum_{n=1}^{N} \Big(\varphi_j\, n_i\, D \Big)_n \end{aligned} \tag{6.28}$$

Here the half breadth $b\,(= B/2)$ on the still water plane is taken as the representative length for normalization, and symbol ϵ_j means that $\epsilon_j = 1$ for $j = 1$, 2 and $\epsilon_j = b$ for $j = 3$. Moreover n in (6.28) denotes the sequential number of segments on the body surface, D is the length of each segment defined in (6.17), and the component of normal vector n_i is also constant on each segment.

The wave-exciting force in the diffraction problem can be computed in nondimensional form by the following:

$$E'_i = \frac{E_i}{\rho g \zeta_a b \epsilon_i} = \int_{S_H} \varphi_D(x,y) n_i\, ds = \sum_{n=1}^{N} \Big(\varphi_D\, n_i\, D \Big)_n \tag{6.29}$$

The Kochin function is, as defined by (3.13) and (3.22), the complex wave-amplitude function representing effects of the body geometry and the mode of motion on generated waves. This Kochin function was defined in the form

$$H_j^{\pm} = \int_{S_H} \left(\frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial}{\partial n} \right) e^{-K\eta \pm iK\xi}\, ds \quad (j = 1 \sim 3) \tag{6.30}$$

$$H_4^{\pm} = -\int_{S_H} \varphi_D\, \frac{\partial}{\partial n}\, e^{-K\eta \pm iK\xi}\, ds \tag{6.31}$$

Let us consider a numerical calculation method for these equations. Recalling that the body boundary condition in the radiation problem was given as $\partial \varphi_j / \partial n = n_j$ (which is constant on each segment) and that the function conjugate to $e^{-K\eta \pm iK\xi}$ is given by $\mp i\, e^{-K\eta \pm iK\xi}$, we can obtain the calculation formula for the Kochin function as follows:

$$H_j^{\pm} = \sum_{n=1}^{N} \Big\{ \big(n_j \big)_n \mathcal{F}_n^{\pm} - \big(\varphi_j \big)_n \mathcal{G}_n^{\pm} \Big\} \quad (j = 1 \sim 3) \tag{6.32}$$

$$H_4^{\pm} = -\sum_{n=1}^{N} \big(\varphi_D \big)_n \mathcal{G}_n^{\pm} \tag{6.33}$$

where

$$\mathcal{F}_n^{\pm} = \int_{s_n} e^{-K\eta \pm iK\xi}\, ds = \mp \frac{i}{K}\, e^{\mp i\delta} \Big[e^{-K\eta \pm iK\xi} \Big]_n^{n+1} \tag{6.34}$$

$$\mathcal{G}_n^{\pm} = \int_{s_n} \frac{\partial}{\partial n}\, e^{-K\eta \pm iK\xi}\, ds = \mp i \Big[e^{-K\eta \pm iK\xi} \Big]_n^{n+1} \tag{6.35}$$

Details of analytical integration of the exponential function shown in (6.34) on each segment can be seen in page 52 of the reference book introduced in a footnote.

Whether computed values of hydrodynamic forces and the Kochin function are correct or not may be confirmed by checking various relations that are proven theoretically. As examples of those relations, the energy conservation associated with the damping force and the Haskind-Newman relation for the wave-exciting force can be expressed in nondimensional form as follows:

$$B'_{ij} = H_i^+ \overline{H}_j^+ \,, \qquad E'_i = H_i^+ \tag{6.36}$$

Furthermore, the relation between the Kochin functions in the radiation and diffraction problems is expressed from (3.18) and (3.87) as follows:

$$\left.\begin{aligned} H_4^{\pm} &= i\, e^{i\varepsilon_2} \cos\varepsilon_2 \mp e^{i\varepsilon_j} \sin\varepsilon_j \\ \varepsilon_j &= \arg\big(-i\, H_j^+ \big) \end{aligned}\right\} \tag{6.37}$$

6.5 Motion Equations of a Floating Body

The motion equations of a floating body will be considered in terms of the coordinate system with the origin taken at the center of gravity G. The analyses so far have been made using the coordinate system with the origin on the calm water surface $y = 0$, and thus let us consider mutual relations first. With assumption that the center of gravity G is located just below the origin O at $y = 0$ with vertical distance $\ell_G \equiv \overline{OG}$, the relations between the motion amplitudes are given as

$$X_1 = X_1^G + \ell_G X_3^G \,, \quad X_2 = X_2^G \,, \quad X_3 = X_3^G \tag{6.38}$$

where X_j^G denotes the complex amplitude of the j-th mode of motion at the center of gravity. On the other hand, the outer product of the normal vector and the position vector from the center of gravity for roll motion is given by

$$n_3^G = y n_2 - (z - \ell_G) n_1 = n_3 + \ell_G n_1 \tag{6.39}$$

Thus we can see that the velocity potential in roll around the center of gravity can be given as

$$\varphi_3^G = \varphi_3 + \ell_G \varphi_1 \tag{6.40}$$

By using (6.39) and (6.40), the radiation forces measured at the center of gravity can be computed with transfer function T_{ij} defined by (3.31), in the form

$$F_1^G = T_{11} X_1^G + T_{12} X_2^G + \big(T_{13} + \ell_G T_{11}\big) X_3^G \tag{6.41}$$

$$F_2^G = T_{21} X_1^G + T_{22} X_2^G + \big(T_{23} + \ell_G T_{21}\big) X_3^G \tag{6.42}$$

$$F_3^G = \big(T_{31} + \ell_G T_{11}\big) X_1^G + \big(T_{32} + \ell_G T_{12}\big) X_2^G + \Big\{ \big(T_{33} + \ell_G T_{13}\big) + \ell_G \big(T_{31} + \ell_G T_{11}\big) \Big\} X_3^G \tag{6.43}$$

Likewise the wave-exciting forces measured at the center of gravity can be computed from

$$E_1^G = E_1 \,, \quad E_2^G = E_2 \,, \quad E_3^G = E_3 + \ell_G E_1 \tag{6.44}$$

In addition to hydrodynamic forces described above, we need to include the restoring force which can be computed by integrating the variance in the hydrostatic pressure due to displacement of a body. Then we can establish the coupled motion equations among sway, heave, and roll for an asymmetric general-shaped body. Since the fluid forces are given in nondimensional form like (6.28) and (6.29), the motion equations are also nondimensionalized in terms of the half breadth b, and the results may be expressed in the form

$$-\left(m' + Z'_{11}\right)\frac{X_1^G}{\zeta_a} - Z'_{12}\frac{X_3^G}{\zeta_a} - \left(Z'_{13} + \ell'_G Z'_{11}\right)\frac{X_3^G b}{\zeta_a} = \frac{E'_1}{Kb} \tag{6.45}$$

$$-Z'_{21}\frac{X_1^G}{\zeta_a} - \left(m' + Z'_{22} - \frac{C'_{22}}{Kb}\right)\frac{X_2^G}{\zeta_a} - \left(Z'_{23} + \ell'_G Z'_{21}\right)\frac{X_3^G b}{\zeta_a} = \frac{E'_2}{Kb} \tag{6.46}$$

$$\begin{aligned}&-\left(Z'_{31} + \ell'_G Z'_{11}\right)\frac{X_1^G}{\zeta_a} - \left(Z'_{32} + \ell'_G Z'_{12}\right)\frac{X_2^G}{\zeta_a}\\ &\quad -\left\{m'\kappa'^2_{xx} + Z'_{33} + \ell'_G Z'_{13} + \ell'_G\left(Z'_{31} + \ell'_G Z'_{11}\right) - \frac{C'_{33}}{Kb}\right\}\frac{X_3^G b}{\zeta_a} = \frac{1}{Kb}\left(E'_3 + \ell'_G E'_1\right)\end{aligned} \tag{6.47}$$

where m' and κ'_{xx} denote the nondimensional mass of a body and the gyrational radius in roll, respectively. Including these, the prime means nondimensional values defined as follows:

$$\left.\begin{aligned}&m' = \frac{\rho\nabla}{\rho b^2} = \frac{\nabla}{b^2}\,, \quad \kappa'_{xx} = \frac{\kappa_{xx}}{b}\,, \quad \ell'_G = \frac{\ell_G}{b},\\ &Z'_{ij} = A'_{ij} - i\,B'_{ij}\,, \quad C'_{22} = \frac{C_{22}}{\rho g b} = \frac{B}{b},\\ &C'_{33} = \frac{C_{33}}{\rho g b^3} = m'\,\frac{\overline{GM}}{b}\end{aligned}\right\} \tag{6.48}$$

It should be noted that $\overline{GM}$ can be calculated with $\ell_G = \overline{OG}$ once the body geometry is given, and the displacement volume (sectional area in 2D) ∇ and the distance to the center of buoyancy (center of sectional area) $\overline{OB}$ can be calculated as well only with the body geometry. Therefore what is needed as input data are the center of gravity and the gyrational radius in roll.

Appendix

A 1 Numerical Computation Method for Free-surface Green Function

It has been shown that the free-surface Green function (the velocity potential due to periodic source with unit strength) is expressed as (2.31)–(2.33). However, for actual numerical computations, we must consider how to treat the integral with respect to variable k. In this section, necessary mathematical transformation will be shown for accurate and efficient numerical computations.

First, the singular integral part in the Green function, denoted as I_1, can be expressed from (2.23) and (2.30) in the form

$$I_1 = \oint_0^\infty \frac{e^{-ky}\cos kx}{k-K}\,dk - i\pi e^{-Ky}\cos Kx \tag{A-1}$$

$$= \int_0^\infty \frac{k\cos ky - K\sin ky}{k^2+K^2}\,e^{-k|x|}\,dk - i\pi e^{-Ky-iK|x|} \tag{A-2}$$

Denoting the integral appearing in (A-2) as

$$F(x,y) = \int_0^\infty \frac{k\cos ky - K\sin ky}{k^2+K^2}\,e^{-k|x|}\,dk, \tag{A-3}$$

we can see that this function satisfies the following differential equation

$$\frac{dF}{dy} + KF = -\int_0^\infty e^{-k|x|}\sin ky\,dk = -\frac{y}{x^2+y^2}. \tag{A-4}$$

Therefore its solution can be obtained as follows:

$$F = e^{-Ky}\left[-\int_0^y \frac{\eta}{x^2+\eta^2}\,e^{K\eta}\,d\eta + C\right], \tag{A-5}$$

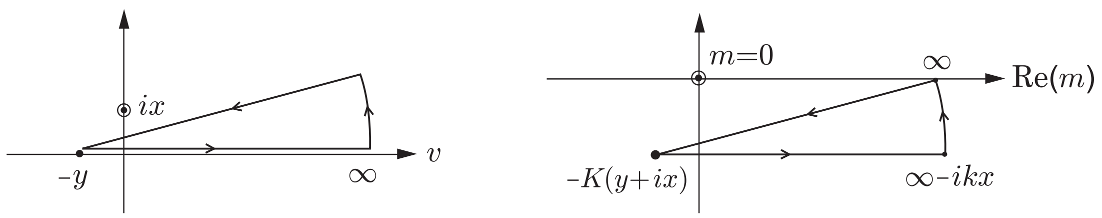
where C in the unknown coefficient in a homogeneous solution of the differential equation, but it can be determined by considering the value at $y=0$, i.e. $C = F(x,0)$. Thus from (A-3) it may be explicitly given as

$$C = \int_0^\infty \frac{k}{k^2+K^2}\,e^{-k|x|}\,dk = -\int_{-\infty}^0 \frac{\eta}{x^2+\eta^2}\,e^{K\eta}\,d\eta \tag{A-6}$$

Substituting this result, (A-5) can be written and transformed further as follows:

$$F = e^{-Ky}\left[-\int_{-\infty}^y \frac{\eta}{x^2+\eta^2}\,e^{K\eta}\,d\eta\right]$$

$$= e^{-Ky}\int_{-y}^\infty \frac{v}{x^2+v^2}\,e^{-Kv}\,dv = e^{-Ky}\,\mathrm{Re}\int_{-y}^\infty \frac{e^{-Kv}}{v-ix}\,dv \tag{A-7}$$



Integration path in the complex plane for $x > 0$

With variable transformation of $K(v - ix) = m$, the integration path in the complex m-plane may be taken as shown in the figure above. Since there is no singularity inside the closed integration path, we have the following expression by virtue of the residue theorem:

$$\left.\begin{aligned} F &= \mathrm{Re}\left[e^{-Ky-iKx} \int_{-K(y+ix)}^{\infty} \frac{e^{-m}}{m}\, dm \right] = \mathrm{Re}\left[e^{-Kz} E_1(-Kz) \right] \\ E_1(\zeta) &= \int_{\zeta}^{\infty} \frac{e^{-m}}{m}\, dm, \quad z = y + ix \end{aligned}\right\} \tag{A-8}$$

Here $E_1(\zeta)$ denotes the exponential integral function with complex variable; its series expansion and asymptotic expansion are well studied and summarized in Appendix A 2, with which we can perform the fast computation with desired accuracy.

The transformation for (A-8) has been done with assumption of $x > 0$, but (A-8) is valid also for $x < 0$, because the integration path in the v-plane should be taken below the real axis (in the 4th quadrant) and again no singularity exists inside the closed path.

Substituting (A-8) for $F(x, y)$ defined by (A-3) into (A-2), we have the final result in the form

$$I_1 = \mathrm{Re}\left[e^{-Kz} E_1(-Kz) \right] - i\pi e^{-Ky-iK|x|} \tag{A-9}$$

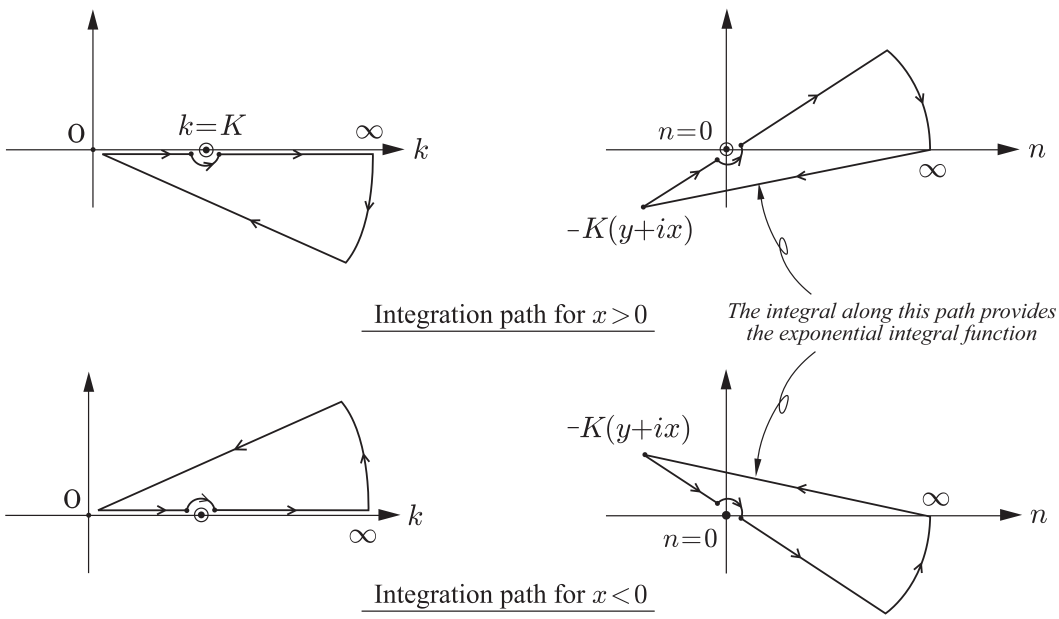
Here, by writing $E_1(-Kz)$ in the form

$$E_1(-Kz) = E_C + iE_S \tag{A-10}$$

we can transform I_1 as follows:

$$\begin{aligned} I_1 &= e^{-Ky}\,\mathrm{Re}\left[e^{-Kx}\left(E_C + iE_S \right)\right] - \pi e^{-Ky}\left\{ \mathrm{sgn}(x) \sin Kx + i \cos Kx \right\} \\ &= e^{-Ky}\left[E_C \cos Kx + \left\{ E_S - \pi\, \mathrm{sgn}(x) \right\} \sin Kx \right] - i\pi e^{-Ky} \cos Kx \qquad \text{(A-11)} \\ &= \mathrm{Re}\left[e^{-Kz}\left\{ E_1(-Kz) - i\pi\, \mathrm{sgn}(x) \right\}\right] - i\pi e^{-Ky} \cos Kx, \qquad \text{(A-12)} \end{aligned}$$

where $z = y + ix$.



It is noteworthy that the final result (A-9) or (A-11) is expedient for numerical computations, but its derivation might be not smart. Thus we will show that (A-12) can be obtained directly from (A-1).

First we note that (A-1) can be written in the form

$$I_1 = \mathrm{Re} \oint_0^{\infty} \frac{e^{-k(y+ix)}}{k-K}\, dk - i\pi e^{-Ky} \cos Kx \tag{A-13}$$

Therefore, comparing (A-13) with (A-12), we can expect the following result:

$$\oint_0^{\infty} \frac{e^{-k(y+ix)}}{k-K}\, dk = e^{-KZ} \{ E_1(-Kz) - i\pi \,\mathrm{sgn}(x) \} \tag{A-14}$$

This result can be proven by the variable transformation $(k-K)(y+ix) = n$ and deformation of the integration path in the complex plane. As shown in the figure above, for $x > 0$ the integration path in the k-plane should be taken in the 4th quadrant, and reversely for $x < 0$ it should be taken in the 1st quadrant for ensuring the convergence along an arc at infinity. Then, through the variable transformation, we can see the corresponding integration path in the n-plane is taken as shown in the figure above. We note that the direction of integration along half a small circle around the singular point at $n = 0$ is opposite depending on $x > 0$ or $x < 0$. Therefore by virtue of the residue theorem and Cauchy's integral theorem, we can obtain the following result:

$$\oint_0^{\infty} \frac{e^{-k(y+ix)}}{k-K}\, dk + i\pi \mathrm{sgn}(x)\, e^{-K(y+ix)} - e^{-K(y+ix)} \int_{-K(y+ix)}^{\infty} \frac{e^{-n}}{n}\, dn = 0 \tag{A-15}$$

Namely

$$\oint_0^{\infty} \frac{e^{-k(y+ix)}}{k-K}\, dk = e^{-Kz} \{ E_1(-Kz) - i\pi \mathrm{sgn}(x) \} \tag{A-16}$$

$$\text{where} \qquad z = y + ix$$

We can confirm that (A-16) provides us with the same result for the integral I_1 as (A-12).

Summarizing above, the variable transformation $(k-K)(y+ix) = n$ applied to (A-1) is efficient for the purpose of proving the final result of (A-9). However, mathematical transformation from (A-2) might be useful and educational in extending the present treatment to 3D problems.

A 2 Numerical Computation for Exponential Integral Function

Details for mathematical derivation are omitted, but it is known that the exponential integral function with complex argument can be expressed in several ways as written below. By combining these expressions appropriately, the exponential integral function can be computed for all values of complex variable $z = x + iy$ with high acuracy and efficiency.

(1) Series Expansion

$$E_1(z) = \gamma - \log z - \sum_{n=1}^{\infty} \frac{(-z)^n}{n \cdot n!}\,, \tag{A-17}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant.

(2) Continued Fraction

$$e^z E_1(z) = \cfrac{1}{z + \cfrac{1}{1 + \cfrac{1}{z + \cfrac{2}{1 + \cfrac{2}{z + \cfrac{3}{1 + \cfrac{3}{z + \cdots}}}}}}} \tag{A-18}$$

(3) Asymptotic Expansion

$$e^z E_1(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{n!}{(-z)^n} \,. \tag{A-19}$$

A 3 Lewis-Form Approximation

As shown in Fig. A-1, the conformal mapping of a real body (in the physical plane of $z = x + iy$) onto a circle with unit radius (in the transformed plane of $\zeta = \xi + i\eta$) is considered. In terms of the expansion-contraction coefficient (scale factor) M and two more coefficients a_1 and a_3, the equation for this conformal mapping can be written in the form

$$x + iy = M\left\{ \zeta + \frac{a_1}{\zeta} + \frac{a_3}{\zeta^3} \right\}. \tag{A-20}$$

Since the body surface is defined as $r = 1$ (the radius equal to 1), $\zeta = \sin\theta + i\cos\theta = i\,e^{-i\theta}$ is substituted in the above. Then the coordinates (x, y) can be expressed with θ in the form

$$\left.\begin{aligned} x &= M\Big\{ (1 + a_1)\sin\theta - a_3 \sin 3\theta \Big\} \\ y &= M\Big\{ (1 - a_1)\cos\theta + a_3 \cos 3\theta \Big\} \end{aligned}\right\} \tag{A-21}$$

The body shape represented by (A-21) is called Lewis form.

The unknowns are M, a_1, and a_3, which may be determined by specifying the following three quantities:

1) Half breadth $\quad B/2 = M(1 + a_1 + a_3),$ (A-22)

2) Draft $\quad d = M(1 - a_1 + a_3),$ (A-23)

3) Sectional area $\quad S = \frac{\pi}{2} M^2 (1 - a_1^2 - 3a_3^2)$ (A-24)

The typical procedure for determining M, a_1, and a_3 is as follows. First, nondimensional parameters, the half-breadth-to-draft ratio H_0 and the sectional area ratio σ, are defined as follows:

$$H_0 = \frac{B/2}{d} = \frac{1 + a_1 + a_3}{1 - a_1 + a_3}, \tag{A-25}$$

$$\sigma = \frac{S}{Bd} = \frac{\pi}{4} H_0 \frac{1 - a_1^2 - 3a_3^2}{(1 + a_1 + a_3)^2} \,. \tag{A-26}$$

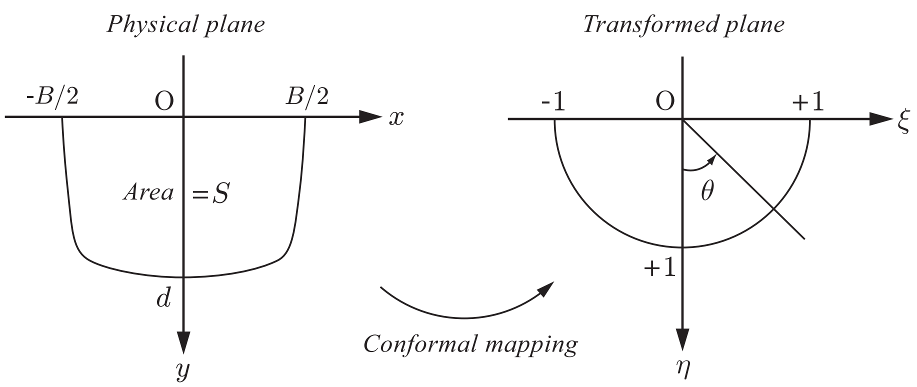


Fig.A-1 Coordinate system in the Lewis-form approximation.

From (A-22) and (A-23), we can obtain

$$a_1 = \frac{H_0 - 1}{2(M/d)}\,, \quad a_3 = \frac{H_0 + 1}{2(M/d)} - 1 \tag{A-27}$$

Then substituting these in (A-26), a quadratic equation for M/d will be obtained. Selection of the correct solution from two possible solutions can be done with physical argument by considering a special case. Namely a flat plate (which is described by $a_1 = -1$ and $a_3 = 0$, resulting from $B/2 = 0$, $H_0 = \sigma = 0$) is considered. In this case, we can see that M/d must be equal to $1/2$. From this consideration, it follows that

$$\frac{M}{d} = \frac{3(H_0+1) - \sqrt{(H_0+1)^2 + 8H_0(1 - 4\sigma/\pi)}}{4} \tag{A-28}$$

On the other hand, the coordinates for the contour of body shape can be given in nondimensional form as follows:

$$\left.\begin{aligned} x' &= \frac{x}{B/2} = \frac{1}{H_0}\left(\frac{M}{d}\right)\Big\{(1+a_1)\sin\theta - a_3 \sin 3\theta\Big\} \\ y' &= \frac{y}{B/2} = \frac{1}{H_0}\left(\frac{M}{d}\right)\Big\{(1-a_1)\cos\theta + a_3 \cos 3\theta\Big\} \end{aligned}\right\} \tag{A-29}$$

To sum up, M/d is computed first from (A-28) in terms of H_0 and σ, then a_1 and a_3 are computed from (A-27), and finally the nondimensional coordinates (x', y') for the body shape are computed from (A-29).

Source program for a general-shaped 2D floating body

```
C MAIN FILE NAME:( OEPANEL/IEM2D.F )
C     ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     +          2-D RADIATION AND DIFFRACTION PROBLEMS            +
C     +               OF A GENERAL-SHAPED 2-D BODY                 +
C     +                                                            +
C     +               BY INTEGRAL-EQUATION METHOD                  +
C     +          WITH THERAPY FOR IRREGULAR FREQUENCIES            +
C     +                                                            +
C     +           ADDED-MASS AND DAMPING COEFFICIENTS,             +
C     +         WAVE EXCITING FORCE, WITH ACCURACY CHECK           +
C     +              SWAY, HEAVE, & ROLL MOTIONS AND               +
C     +          TRANSMISSION & REFLECTION COEFFICIENTS            +
C     +                                                            +
C     +                     CODED BY M. KASHIWAGI ON 1994 8/18     +
C     +                     MODIFIED FOR OE-PANEL ON 2001 9/21     +
C     +                     AT R.I.A.M. KYUSHU UNIVERSITY          +
C     ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      IMPLICIT DOUBLE PRECISION (A-H,K,O-Z)
      COMMON /PAI/ PI,PI05,PI2
C
      PI  =3.14159265358979D0
      PI05=PI*0.5D0
      PI2 =PI*2.0D0
      IPRINT=1
      NPRINT=0
C
C **********************( INPUT DATA )**************************
C       NB   : NUMBER OF PANELS OVER SUBMERGED BODY (MAX=100)
C       H0   : RATIO OF HALF-BREADTH TO DRAFT  (=B/2/D)
C       SIGMA: SECTIONAL AREA RATIO            (=S/B/D)
C
C       OGD  : CENTER OF GRAVITY / DRAFT        =OG/D
C       KZZB : GYRATIONAL RADIUS / HALF-BREADTH =KZZ/(B/2)
C
C       AKB  : NONDIMENSIONAL WAVENUMBER =W*W/G*(B/2)
C **************************************************************
C
      READ(5,*)    NB,H0,SIGMA
      WRITE(6,600) NB,H0,SIGMA
      OGD =0.05D0
      KZZB=0.35D0
C
        NT=NB+3
        CALL OFFSET(NB,NT,H0,SIGMA,OGD,KZZB,NPRINT)
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
    1 READ(5,*,END=9) AKB
C     /
      CALL SOLVE (NB,NT,AKB)
      CALL KOCHIN(NB,AKB,IPRINT)
      CALL FORCE (NB,AKB,IPRINT)
      CALL MOTION(AKB,IPRINT)
      CALL TRCOEF(AKB,IPRINT)
C     /
      GOTO 1
    9 STOP
  600 FORMAT(//14X,48('*')
     &   /19X,'2-D RADIATION AND DIFFRACTION PROBLEMS',
     &   /19X,'    OF A GENERAL-SHAPED 2-D BODY',
     &   /19X,'     BY INTEGRAL-EQUATION METHOD',/14X,48('*'),
     &  //15X,'NUMBER OF PANELS OVER WHOLE BODY (NB)=',I4,
     &   /15X,'HALF-BEAM TO DRAFT RATIO   H0(=B/2/D)=',F8.4,
     &   /15X,'SECTIONAL AREA RATIO    SIGMA(=S/B/D)=',F8.4/)
      END
C *****************************************************************
C **             OFFSET DATA FOR THE LEWIS FORM SHIP             **
C **    THIS SUBROUTINE ASSUMES THE BODY GEOMETRY TO BE SYMMETRIC    **
C *****************************************************************
      SUBROUTINE OFFSET(NB,NT,H0,SIGMA,OGD,KZZB,IPRINT)
      IMPLICIT DOUBLE PRECISION (A-H,K,O-Z)
C
      PARAMETER (MX=105,NP=100,NQ=101)
      COMMON /PAI/ PI,PI05,PI2
      COMMON /MDT/ CMAS,C22,OG,KZZ,GM
      COMMON /ELM/ XP(MX),YP(MX),XQ(NQ),YQ(NQ)
      COMMON /VN2/ VN(3,NP)
```

```
C
      IAD=NT-NB
C     /
      RSUB=(H0+1.0D0)**2+8.0D0*H0*(1.0D0-4.0D0*SIGMA/PI)
      AMD =0.25D0*(3.0D0*(H0+1.0D0)-DSQRT(RSUB))
      A1  =0.5D0*(H0-1.0D0)/AMD
      A3  =0.5D0*(H0+1.0D0)/AMD-1.0D0
      AMB =AMD/H0
C     /
      DTH=PI/DFLOAT(NB)
      DO 100 J=1,NB+1
      TH=PI05-DTH*DFLOAT(J-1)
      XQ(J)=AMB*((1.0D0+A1)*DSIN(TH)-A3*DSIN(3.0D0*TH))
      YQ(J)=AMB*((1.0D0-A1)*DCOS(TH)+A3*DCOS(3.0D0*TH))
  100 CONTINUE
C     /
      DO 110 I=1,NB
      XP(I)=(XQ(I+1)+XQ(I))/2.0D0
      YP(I)=(YQ(I+1)+YQ(I))/2.0D0
      DX=XQ(I+1)-XQ(I)
      DY=YQ(I+1)-YQ(I)
      D =DSQRT(DX*DX+DY*DY)
      VN(1,I)= DY/D
      VN(2,I)=-DX/D
      VN(3,I)=XP(I)*VN(2,I)-YP(I)*VN(1,I)
  110 CONTINUE
C     /
      IF(IAD.EQ.0) GOTO 130
      DS=(XQ(1)-XQ(NB+1))/DFLOAT(IAD+1)
      DO 120 I=1,IAD
      II=NB+I
      XP(II)=XQ(NB+1)+DS*DFLOAT(I)
      YP(II)=0.0D0
  120 CONTINUE
C     /
  130 CMAS=2.0D0*SIGMA/H0
      C22 =(XQ(1)-XQ(NB+1))/XQ(1)
      OG  =OGD/H0
      KZZ =KZZB
         SUM=0.0D0
         DO 200 J=1,NB
         S1 =YQ(J+1)-YQ(J)
         S2 =XQ(J  )*(2.0D0*YQ(J  )+YQ(J+1))
         S3 =XQ(J+1)*(2.0D0*YQ(J+1)+YQ(J  ))
         SUM=SUM+S1*(S2+S3)
  200    CONTINUE
      OBM=SUM/6.0D0
      GM =(2.0D0/3.0D0-OBM)/CMAS+OG
C     /
      WRITE(6,600) CMAS,C22,OGD,KZZ,GM
      IF(IPRINT.EQ.0) RETURN
      WRITE(6,610)
      DO 300 J=1,NB+1
  300 WRITE(6,620) J,XQ(J),YQ(J),XP(J),YP(J)
  600 FORMAT(
     &    15X,'NONDIMENSIONAL MASS------- S/(B/2)**2=',F8.5,
     &   /15X,'HEAVE RESTORING FORCE COEFF--AW/(B/2)=',F8.5,
     &   /15X,'CENTER OF GRAVITY-----------------OG/D=',F8.5,
     &   /15X,'GYRATIONAL RADIUS-----------KZZ/(B/2)=',F8.5,
     &   /15X,'METACENTRIC HEIGHT-----------GM/(B/2)=',F8.5/)
  610 FORMAT(/15X,'***** CHECK OF ORDINATES *****'
     &   /8X,'J',6X,'XQ',8X,'YQ',10X,'XP',8X,'YP')
  620 FORMAT(7X,I2,1X,2F10.5,2X,2F10.5)
      RETURN
      END
C ***********************************************************************
C **      INFLUENCE COEFFICIENTS DUE TO LOG-TYPE SINGULAR TERMS       **
C ***********************************************************************
      SUBROUTINE SDSUB(XPI,YPI,NB,SS,DD)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
      PARAMETER (MX=105,NQ=101)
      DIMENSION SS(NB),DD(NB)
      COMMON /ELM/ XP(MX),YP(MX),XQ(NQ),YQ(NQ)
C
      DO 100 J=1,NB
      SWA=0.0D0
```

```
      DWA=0.0D0
      IF(DABS(YPI).LT.1.0D-8) GOTO 10
        DX=XQ(J+1)-XQ(J)
        DY=YQ(J+1)-YQ(J)
        D =DSQRT(DX*DX+DY*DY)
        CDEL=DX/D
        SDEL=DY/D
      XA=XPI-XQ(J  )
      XB=XPI-XQ(J+1)
C     /
      SL=-1.0D0
      DO 200 L=1,2
      SL=-SL
      YA=SL*YPI-YQ(J  )
      YB=SL*YPI-YQ(J+1)
      SUBA=XA*CDEL+YA*SDEL
      SUBB=XB*CDEL+YB*SDEL
      COEF=XA*SDEL-YA*CDEL
      ABSC=DABS(COEF)
        WA1=0.5D0*(SUBB*DLOG(XB*XB+YB*YB)-SUBA*DLOG(XA*XA+YA*YA))
        IF(ABSC.LT.1.0D-10) THEN
        WA2=0.0D0
        WA3=0.0D0
        ELSE
        WA2=ABSC*(DATAN(SUBB/ABSC)-DATAN(SUBA/ABSC))
        WA3=WA2/COEF
        ENDIF
      SWA=SWA-(WA1+WA2)*SL
      DWA=DWA+ WA3*SL
  200 CONTINUE
C     /
   10 SS(J)=SWA
      DD(J)=DWA
  100 CONTINUE
      RETURN
      END
C ********************************************************************
C **      INFLUENCE COEFFICIENTS DUE TO FREE-SURFACE WAVE TERM       **
C ********************************************************************
      SUBROUTINE SDCAL(XPI,YPI,AK,NB,ZS,ZD)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      PARAMETER (MX=105,NQ=101)
      DIMENSION ZS(NB),ZD(NB)
      COMMON /PAI/ PI,PI05,PI2
      COMMON /ELM/ XP(MX),YP(MX),XQ(NQ),YQ(NQ)
C
      Z0=(0.0D0,0.0D0)
      ZI=(0.0D0,1.0D0)
      DO 100 J=1,NB
      ZS(J)=Z0
      ZD(J)=Z0
  100 CONTINUE
C     /
C+-+-+-+-+-+-+-+ INITIALIZATION +-+-+-+-+-+-+-+
      XX=XPI-XQ(1)
      YY=YPI+YQ(1)
      SGNX=DSIGN(1.0D0,XX)
      IF(DABS(XX).LT.1.0D-10) SGNX=0.0D0
        XE=-AK*YY
        YE=-AK*DABS(XX)
        ZETA=DCMPLX(XE,YE)
        CALL EZE1Z(XE,YE,EC,ES)
      RFL1=0.5D0*DLOG(XX**2+YY**2)
      RFT1=DATAN2(YY,XX)
      ZFC1=  EC-PI*CDEXP(ZETA)*ZI
      ZFS1=(-ES+PI*CDEXP(ZETA))*SGNX
C+-+-+-+-+-+-+-+-+-+-++-+-+-+-+-+-+-+-+-+-+-+-+
      DO 200 J=1,NB
      XX=XPI-XQ(J+1)
      YY=YPI+YQ(J+1)
      SGNX=DSIGN(1.0D0,XX)
      IF(DABS(XX).LT.1.0D-10) SGNX=0.0D0
        XE=-AK*YY
        YE=-AK*DABS(XX)
        ZETA=DCMPLX(XE,YE)
```

```
        CALL EZE1Z(XE,YE,EC,ES)
      RFL2=0.5D0*DLOG(XX**2+YY**2)
      RFT2=DATAN2(YY,XX)
      ZFC2=  EC-PI*CDEXP(ZETA)*ZI
      ZFS2=(-ES+PI*CDEXP(ZETA))*SGNX
C     /
      DX=XQ(J+1)-XQ(J)
      DY=YQ(J+1)-YQ(J)
      D =DSQRT(DX*DX+DY*DY)
      CDEL=DX/D
      SDEL=DY/D
      SUB =SDEL*(RFL2-RFL1)+CDEL*(RFT2-RFT1)
      ZSUB=SDEL*(ZFC2-ZFC1)+CDEL*(ZFS2-ZFS1)
      ZS(J)=ZS(J)+2.0D0/AK*(SUB+ZSUB)
      ZD(J)=ZD(J)+2.0D0*(ZFS2-ZFS1)
      RFL1=RFL2
      RFT1=RFT2
      ZFC1=ZFC2
      ZFS1=ZFS2
  200 CONTINUE
      RETURN
      END
C **********************************************************************
C **     SOLUTION OF INTEGRAL EQUATION FOR THE VELOCITY POTENTIAL     **
C **         INCLUDING ELIMINATION OF IRREGULAR FREQUENCIES           **
C **********************************************************************
      SUBROUTINE SOLVE(NB,NT,AK)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      PARAMETER (MX=105,NP=100,NQ=101,NEQ=4,SML=1.0D-15)
      DIMENSION ZSA(MX,NP),ZSB(MX,NEQ),ZAA(NP,NP),ZBB(NP,NEQ)
      DIMENSION ZS(NP),ZD(NP),SS(NP),DD(NP)
C
      COMMON /PAI/ PI,PI05,PI2
      COMMON /ELM/ XP(MX),YP(MX),XQ(NQ),YQ(NQ)
      COMMON /VN2/ VN(3,NP)
      COMMON /FAI/ ZFI(4,NP)
C
      Z0=(0.0D0,0.0D0)
      ZI=(0.0D0,1.0D0)
      DO 10 I=1,NB
      DO 20 J=1,NB
   20 ZAA(I,J)=Z0
      DO 10 M=1,NEQ
      ZBB(I,M)=Z0
   10 CONTINUE
C     /
      DO 30 I=1,NT
      DO 40 J=1,NB
   40 ZSA(I,J)=Z0
      DO 50 M=1,NEQ
   50 ZSB(I,M)=Z0
      IF(I.LE.NB) ZSA(I,I)=DCMPLX(PI,0.0D0)
   30 CONTINUE
C
      DO 100 I=1,NT
      CALL SDSUB(XP(I),YP(I),NB,SS,DD)
      CALL SDCAL(XP(I),YP(I),AK,NB,ZS,ZD)
C +-+-+-+-+-+-+-+-+-( LEFT-HAND SIDE )-+-+-+-+-+-+-+-+-+-+-+
      DO 110 J=1,NB
      ZSA(I,J)=ZSA(I,J)+DD(J)+ZD(J)
  110 CONTINUE
C +-+-+-+-+-+-+-+-+-( RIGHT-HAND SIDE )+-+-+-+-+-+-+-+-+-+-+
      DO 120 M=1,3
      DO 120 J=1,NB
      ZSB(I,M)=ZSB(I,M)+(SS(J)+ZS(J))*VN(M,J)
  120 CONTINUE
      ZSB(I,4)=PI2*CDEXP(-AK*(YP(I)-ZI*XP(I)))
  100 CONTINUE
C
C ++++++++++++ LEAST-SQUARES METHOD ++++++++++++++++
      DO 200 I=1,NB
      DO 210 J=1,NB
      DO 210 K=1,NT
      ZAA(I,J)=ZAA(I,J)+ZSA(K,I)*ZSA(K,J)
  210 CONTINUE
```

```
      DO 220 M=1,NEQ
      DO 220 K=1,NT
      ZBB(I,M)=ZBB(I,M)+ZSA(K,I)*ZSB(K,M)
  220 CONTINUE
  200 CONTINUE
C ++++++++++++++++++++++++++++++++++++++++++++++++
C
      CALL ZSWEEP(NP,NB,ZAA,ZBB,NEQ,SML)
      IF(CDABS(ZAA(1,1)).LT.SML) WRITE(6,600)
  600 FORMAT(//10X,'*** ERROR: ZSWEEP IN SUBROUTINE (SOLVE)',
     &   ' WAS ABNORMALLY DONE.',/23X,'PLEASE CHECK!'///)
C
      DO 250 M=1,NEQ
      DO 250 I=1,NB
      ZFI(M,I)=ZBB(I,M)
  250 CONTINUE
      RETURN
      END
C *******************************************************************
C **                 CALCULATION OF KOCHIN FUNCTION                 **
C **   WHICH WILL BE USED FOR NUMERICAL CHECK OF VARIOUS RELATIONS   **
C *******************************************************************
      SUBROUTINE KOCHIN(NB,AK,IPRINT)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      PARAMETER (MX=105,NP=100,NQ=101)
      DIMENSION ABAR(3),EPS(3)
      COMMON /PAI/ PI,PI05,PI2
      COMMON /ELM/ XP(MX),YP(MX),XQ(NQ),YQ(NQ)
      COMMON /VN2/ VN(3,NP)
      COMMON /FAI/ ZFI(4,NP)
      COMMON /KCH/ ZHA(4),ZHB(4)
C
      Z0=(0.0D0,0.0D0)
      ZI=(0.0D0,1.0D0)
      DO 10 M=1,4
      ZHA(M)=Z0
      ZHB(M)=Z0
   10 CONTINUE
C     /
      ZETA=-AK*(YQ(1)-ZI*XQ(1))
      ZEOLD=CDEXP(ZETA)
      DO 100 J=1,NB
      DX=XQ(J+1)-XQ(J)
      DY=YQ(J+1)-YQ(J)
      D =DSQRT(DX*DX+DY*DY)
      CDEL=DX/D
      SDEL=DY/D
      ZSUB=-(SDEL+ZI*CDEL)/AK
      ZETA=-AK*(YQ(J+1)-ZI*XQ(J+1))
      ZENEW=CDEXP(ZETA)
      ZFHA =ZSUB*(ZENEW-ZEOLD)
      ZFGA =-ZI*(ZENEW-ZEOLD)
      ZFHB =DCONJG(ZFHA)
      ZFGB =DCONJG(ZFGA)
      ZEOLD=ZENEW
C     /
      DO 110 M=1,3
      ZHA(M)=ZHA(M)+VN(M,J)*ZFHA-ZFI(M,J)*ZFGA
      ZHB(M)=ZHB(M)+VN(M,J)*ZFHB-ZFI(M,J)*ZFGB
  110 CONTINUE
      ZHA(4)=ZHA(4)-ZFI(4,J)*ZFGA
      ZHB(4)=ZHB(4)-ZFI(4,J)*ZFGB
  100 CONTINUE
C     /
      DO 200 I=1,3
      ABAR(I)=AK*CDABS(ZHA(I))
      EPS (I)=DATAN2(-DREAL(ZHA(I)),DIMAG(ZHA(I)))
  200 CONTINUE
      ZSYM=CDEXP(ZI*EPS(2))*DCOS(EPS(2))*ZI
      ZANT=CDEXP(ZI*EPS(1))*DSIN(EPS(1))
      ZHRA=ZSYM-ZANT
      ZHRB=ZSYM+ZANT
C
C ++++++++++++++++++++( PRINT OUT FOR CHECK )++++++++++++++++++++++++
      IF(IPRINT.EQ.0) RETURN
```

```
      RTD=180.0D0/PI
      WRITE(6,600) AK,(ABAR(I),EPS(I)*RTD,I=1,3)
      WRITE(6,610) ZHA(4),ZHRA,ZHB(4),ZHRB
  600 FORMAT(/6X,'******* KOCHIN FUNCTION & ACCURACY CHECK ( ',
     &  'K*B/2=',F8.4,' ) *******',//5X,2(15X,'A-BAR',6X,'EPS(DEG)'),
     &   /9X,'SWAY: ',E12.5,2X,F9.3,4X,'HEAVE: ',E12.5,2X,F9.3,
     &   /9X,'ROLL: ',E12.5,2X,F9.3)
  610 FORMAT(/27X,'DIRECT CALCULATION',7X,'COMPUTED FROM RADIATION',
     &   /5X,'DIFFRACTION (+) ',2E13.4,2X,2E13.4,
     &   /5X,'DIFFRACTION (-) ',2E13.4,2X,2E13.4)
      RETURN
      END
C ******************************************************************
C **   PRESSURE INTEGRAL FOR ADDED-MASS, DAMPING & EXCITING FORCES   **
C **        INCLUDING ACCURACY CHECK OF THE ENERGY CONSERVATION      **
C **                    AND HASKIND-NEWMAN RELATION                  **
C ******************************************************************
      SUBROUTINE FORCE(NB,AK,IPRINT)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      PARAMETER (MX=105,NP=100,NQ=101)
      DIMENSION A(3,3),B(3,3),BE(3,3),EAMP(3),EPHA(3)
C
      COMMON /PAI/ PI,PI05,PI2
      COMMON /ELM/ XP(MX),YP(MX),XQ(NQ),YQ(NQ)
      COMMON /VN2/ VN(3,NP)
      COMMON /FAI/ ZFI(4,NP)
      COMMON /KCH/ ZHA(4),ZHB(4)
      COMMON /FCE/ ZAB(3,3),ZEXF(3)
C
      Z0=(0.0D0,0.0D0)
      ZI=(0.0D0,1.0D0)
      DO 10 I=1,3
      DO 11 J=1,3
   11 ZAB(I,J)=Z0
      ZEXF( I)=Z0
   10 CONTINUE
C
      DO 100 K=1,NB
      DX=XQ(K+1)-XQ(K)
      DY=YQ(K+1)-YQ(K)
      D =DSQRT(DX*DX+DY*DY)
      DO 110 I=1,3
      DO 120 J=1,3
  120 ZAB(I,J)=ZAB(I,J)-ZFI(J,K)*VN(I,K)*D
      ZEXF(I )=ZEXF(I )+ZFI(4,K)*VN(I,K)*D
  110 CONTINUE
  100 CONTINUE
C     /
      DO 150 I=1,3
      DO 160 J=1,3
      A (I,J)= DREAL(ZAB(I,J))
      B (I,J)=-DIMAG(ZAB(I,J))
      BE(I,J)=0.5D0*(ZHA(I)*DCONJG(ZHA(J))+ZHB(I)*DCONJG(ZHB(J)))
  160 CONTINUE
      EAMP(I)=CDABS(ZEXF(I))
      EPHA(I)=DATAN2(DIMAG(ZEXF(I)),DREAL(ZEXF(I)))*180.0D0/PI
  150 CONTINUE
C
C +++++++++++++++++++++++++( PRINT OUT )++++++++++++++++++++++++++++
      IF(IPRINT.EQ.0) RETURN
      WRITE(6,600) NB,AK
      DO 300 I=1,3
      C1=B (I,I)
      C2=BE(I,I)
      CHK=DABS(C1-C2)/DABS(C1+C2)*200.0D0
  300 WRITE(6,610) I,I,A(I,I),B(I,I),BE(I,I),CHK
      WRITE(6,615)
      DO 310 I=1,3
      DO 310 J=1,3
      IF(I.EQ.J) GOTO 310
      WRITE(6,610) I,J,A(I,J),B(I,J),BE(I,J)
  310 CONTINUE
      WRITE(6,630)
      DO 320 I=1,3
      WRITE(6,640) I,ZEXF(I),ZHA(I),EAMP(I),EPHA(I)
```

```
  320 CONTINUE
C     /
  600 FORMAT(//5X,'+++++++ ADDED-MASS & DAMPING COEFF. ( ',
     &    'NB=',I3,', K*B/2=',F8.4,' ) +++++++',//10X,
     &    'I  J',8X,'ADDED-MASS',6X,'DAMPING',9X,
     &    'ENERGY',8X,'ERROR(%)')
  610 FORMAT(8X,'(',I2,',',I2,')',3X,E13.4,3(2X,E13.4))
  615 FORMAT(' ')
  620 FORMAT(8X,'(',I2,',',I2,')',3X,E13.4,2(2X,E13.4))
  630 FORMAT(//5X,'+++++ WAVE EXCITING FORCE +++++',
     &  //17X,'PRESSURE INTEGRAL',13X,'HASKIND-NEWMAN',/9X,'J',
     &   2(7X,'REAL',9X,'IMAG',4X),7X,'AMP',5X,'PHASE(DEG)')
  640 FORMAT(8X,I2,2E13.4,2X,2E13.4,3X,E11.4,2X,F9.3)
      RETURN
      END
C *******************************************************************
C **     CALCULATION OF WAVE-INDUCED MOTIONS (SWAY, HEAVE & ROLL)    **
C **            THE OUTPUT IS ABOUT THE CENTER OF GRAVITY            **
C *******************************************************************
      SUBROUTINE MOTION(AK,IPRINT)
      IMPLICIT DOUBLE PRECISION (A-H,K,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      DIMENSION ZAA(3,3),ZBB(3)
      DIMENSION AMPG(3),PHAG(3),ZMTNG(3)
      COMMON /PAI/ PI,PI05,PI2
      COMMON /MDT/ CMAS,C22,OG,KZZ,GM
      COMMON /FCE/ ZAB(3,3),ZEXF(3)
      COMMON /MTN/ ZMTNO(3)
C
      SML=1.0D-14
C     /
      ZAA(1,1)=-AK*(CMAS+ZAB(1,1))
      ZAA(1,2)=-AK* ZAB(1,2)
      ZAA(1,3)=-AK*(ZAB(1,3)+OG*ZAB(1,1))
      ZBB(1  )= ZEXF(1)
C     /
      ZAA(2,1)=-AK* ZAB(2,1)
      ZAA(2,2)=-AK*(CMAS+ZAB(2,2))+C22
      ZAA(2,3)=-AK*(ZAB(2,3)+OG*ZAB(2,1))
      ZBB(2  )= ZEXF(2)
C     /
      ZAA(3,1)=-AK*(ZAB(3,1)+OG*ZAB(1,1))
      ZAA(3,2)=-AK*(ZAB(3,2)+OG*ZAB(1,2))
      ZAA(3,3)=-AK*(CMAS*KZZ**2+ZAB(3,3)+OG*ZAB(1,3)
     &         +OG*(ZAB(3,1)+OG*ZAB(1,1)))+CMAS*GM
      ZBB(3  )= ZEXF(3)+OG*ZEXF(1)
C     /
C +-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+
      CALL ZSWEEP(3,3,ZAA,ZBB,1,SML)
      IF(CDABS(ZAA(1,1)).LT.SML) WRITE(6,600)
  600 FORMAT(///10X,'+++ ERROR: ZSWEEP IN (MOTION) +++'///)
C +-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+
      DO 100 I=1,3
      ZMTNG(I)=ZBB(I)
  100 CONTINUE
      ZMTNO(1)=ZMTNG(1)+OG*ZMTNG(3)
      ZMTNO(2)=ZMTNG(2)
      ZMTNO(3)=ZMTNG(3)
C     /
      DO 200 I=1,3
      AMPG(I)=CDABS(ZMTNG(I))
      IF(I.EQ.3) AMPG(I)=AMPG(I)/AK
      PHAG(I)=DATAN2(DIMAG(ZMTNG(I)),DREAL(ZMTNG(I)))*180.0D0/PI
  200 CONTINUE
C +++++++++++++++++++++++++( PRINT OUT )+++++++++++++++++++++++++++++
      IF(IPRINT.EQ.0) RETURN
      WRITE( 6,610) AK,(AMPG(I),PHAG(I),I=1,3)
  610 FORMAT(//5X,'+++++ MOTIONS ABOUT ''G''  FOR K*B/2=',F7.3,
     &   ' +++++',/20X,'AMP.',7X,'PHASE',/9X,'SWAY  ',E11.4,
     &   2X,F9.3,' (DEG)',/9X,'HEAVE ',E11.4,2X,F9.3,' (DEG)',
     &   /9X,'ROLL  ',E11.4,2X,F9.3,' (DEG)')
      RETURN
      END
C *******************************************************************
C **          TRANSMISSION AND REFLECTION WAVE COEFFICIENTS          **
C **          INCLUDING NUMERICAL CHECK OF ENERGY RELATION           **
```

```
C **********************************************************************
      SUBROUTINE TRCOEF(AK,IPRINT)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      COMMON /KCH/ ZHA(4),ZHB(4)
      COMMON /MTN/ ZMTN(3)
C
      ZI=(0.0D0,1.0D0)
C     /
      ZTDIF=1.0D0+ZI*ZHB(4)
      ZRDIF=ZI*ZHA(4)
      TT   =CDABS(ZTDIF)
      RR   =CDABS(ZRDIF)
      CDIF =TT**2+RR**2
C     /
      ZTFRE=ZTDIF
      ZRFRE=ZRDIF
      S=1.0D0
      DO 100 I=1,3
      S=-S
      ZTFRE=ZTFRE-ZI*AK*ZMTN(I)*ZHB(I)
      ZRFRE=ZRFRE-ZI*AK*ZMTN(I)*ZHA(I)
  100 CONTINUE
      CT   =CDABS(ZTFRE)
      CR   =CDABS(ZRFRE)
      CFRE =CT**2+CR**2
C     /
C ++++++++++++++++++++++++++( PRINT OUT )++++++++++++++++++++++++++++++
      IF(IPRINT.EQ.0) RETURN
      WRITE(6,600) AK,TT,RR,CDIF,CT,CR,CFRE
  600 FORMAT(//5X,'******* TRANSMISSION & REFLECTION COEFF. ( K*',
     &  'B/2=',F8.4,' ) *******',/29X,'CT',12X,'CR',8X,'CT**2+CR**2',
     &   /10X,'DIFFRACTION',2X,E11.4,3X,E11.4,3X,E12.5,
     &   /10X,'MOTION FREE',2X,E11.4,3X,E11.4,3X,E12.5)
      RETURN
      END
C **********************************************************************
C **              SUBROUTINE OF THE EXPONENTIAL INTEGRAL              **
C **********************************************************************
      SUBROUTINE EZE1Z(XX,YY,EC,ES)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16  (Z)
      DOUBLE PRECISION  NEW
C
      DATA PI,GAMMA/3.14159265358979D0,0.5772156649015D0/
C
      X =XX
      Y =DABS(YY)
      R =DSQRT(X*X+Y*Y)
      C =DATAN2(Y,X)
C
      IF(R.GT.25.0D0)  GO TO 30
      IF(X.GT.0.0D0.AND.R.GT.8.0D0)  GO TO 20
      IF(X.LE.0.0D0.AND.Y.GT.10.0D0) GO TO 20
C++++++++++++ SERIES EXPANSION ++++++++++++++++++++++++++++
      ER=-GAMMA-DLOG(R)+R*DCOS(C)
      EI=-C+R*DSIN(C)
      SB=-R
        DO 100 N=2,100
        FN=DFLOAT(N)
        CN=C*FN
        SB=-SB*R*(FN-1.0D0)/FN/FN
        ER=ER-SB*DCOS(CN)
        EI=EI-SB*DSIN(CN)
        IF(N.EQ.100)  GO TO 1
        IF(EI.EQ.0.0D0)  GO TO 10
        IF(DABS(SB/EI).LE.1.0D-8) GO TO 10
        GO TO 100
   10   IF(DABS(SB/ER).LE.1.0D-8) GO TO 1
  100   CONTINUE
    1 CC=DEXP(X)*DCOS(Y)
      SS=DEXP(X)*DSIN(Y)
      EC=CC*ER-SS*EI
      ES=CC*EI+SS*ER
      IF(YY.LT.0.0D0) ES=-ES
      RETURN
```

```
C+++++++++++++ CONTINUED FRACTION ++++++++++++++++++++++++
   20 Z =DCMPLX(X,Y)
      Z1=(1.0D0,0.0D0)
      ZSUB=(10.0D0,0.0D0)
      ZS  =Z+ZSUB/(Z1+ZSUB/Z)
        DO 200 J=1,9
        ZSUB=DCMPLX(DFLOAT(10-J),0.0D0)
        ZS  =Z+ZSUB/(Z1+ZSUB/ZS)
  200   CONTINUE
      ZSUB=Z1/ZS
      EC=DREAL(ZSUB)
      ES=DIMAG(ZSUB)
      IF(YY.LT.0.0D0) ES=-ES
      RETURN
C++++++++++++ ASYMPTOTIC EXPANSION +++++++++++++++++++++++
   30 OLD=-1.0D0/R
      EXC=OLD*DCOS(C)
      EXS=OLD*DSIN(C)
        DO 300 N=2,100
        NEW=-OLD/R*DFLOAT(N-1)
        IF(EXS.EQ.0.0D0) GO TO 31
        IF(DABS(NEW/EXS).LE.1.0D-8) GO TO 31
        GO TO 32
   31   IF(EXC.EQ.0.0D0) GO TO 32
        IF(DABS(NEW/EXC).LE.1.0D-8) GO TO 33
   32   IF(DABS(OLD).LT.DABS(NEW))  GO TO 33
        OLD=NEW
        EXC=EXC+OLD*DCOS(C*DFLOAT(N))
        EXS=EXS+OLD*DSIN(C*DFLOAT(N))
  300   CONTINUE
   33 EC=-EXC
      ES= EXS
      IF(DABS(PI-DABS(C)).LT.1.0D-10) ES=-PI*DEXP(X)
      IF(YY.LT.0.0D0) ES=-ES
      RETURN
      END
C **********************************************************************
C **      SIMPLE GAUSS SWEEPING METHOD FOR SOLVING COMPLEX MATRIX     **
C **********************************************************************
      SUBROUTINE ZSWEEP(NDIM,N,ZA,ZB,NEQ,EPS)
      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
      IMPLICIT COMPLEX*16 (Z)
C
      DIMENSION ZA(NDIM,NDIM),ZB(NDIM,NEQ)
      DO 5 K=1,N
      P=0.0D0
      DO 1 I=K,N
      IF(P.GE.CDABS(ZA(I,K)))  GO TO 1
      P=CDABS(ZA(I,K))
      IP=I
    1 CONTINUE
      IF(P.LE.EPS) GO TO 6
      IF(IP.EQ.K)  GO TO 7
        DO 2 J=K,N
        ZW=ZA(K,J)
        ZA(K,J)=ZA(IP,J)
    2   ZA(IP,J)=ZW
          DO 20 J=1,NEQ
          ZW=ZB(K,J)
          ZB(K,J)=ZB(IP,J)
   20     ZB(IP,J)=ZW
    7 CONTINUE
      IF(K.EQ.N) GO TO 70
      DO 3 J=K+1,N
    3 ZA(K,J)=ZA(K,J)/ZA(K,K)
   70   DO 30 J=1,NEQ
   30   ZB(K,J)=ZB(K,J)/ZA(K,K)
      DO 5 I=1,N
      IF(I.EQ.K)  GO TO 5
      IF(K.EQ.N)  GO TO 40
        DO 4 J=K+1,N
    4   ZA(I,J)=ZA(I,J)-ZA(I,K)*ZA(K,J)
   40   CONTINUE
          DO 45 J=1,NEQ
   45     ZB(I,J)=ZB(I,J)-ZA(I,K)*ZB(K,J)
    5 CONTINUE
      ZA(1,1)=(1.0D0,0.0D0)
```

```
      RETURN
    6 ZA(1,1)=DCMPLX(DABS(P),0.0D0)
      RETURN
      END
```

```
$ chk
40 1.0 0.8

            ************************************************
                2-D RADIATION AND DIFFRACTION PROBLEMS
                    OF A GENERAL-SHAPED 2-D BODY
                     BY INTEGRAL-EQUATION METHOD
            ************************************************

             NUMBER OF PANELS OVER WHOLE BODY (NB)=  40
             HALF-BEAM TO DRAFT RATIO   H0(=B/2/D)=  1.0000
             SECTIONAL AREA RATIO    SIGMA(=S/B/D)=  0.8000

             NONDIMENSIONAL MASS------- S/(B/2)**2= 1.60000
             HEAVE RESTORING FORCE COEFF--AW/(B/2)= 2.00000
             CENTER OF GRAVITY----------------OG/D= 0.05000
             GYRATIONAL RADIUS-----------KZZ/(B/2)= 0.35000
             METACENTRIC HEIGHT-----------GM/(B/2)= 0.03817

1.0

      ******* KOCHIN FUNCTION & ACCURACY CHECK ( K*B/2=  1.0000 ) *******

                  A-BAR      EPS(DEG)             A-BAR      EPS(DEG)
         SWAY:  0.10974E+01   -16.048    HEAVE:  0.77236E+00    -25.214
         ROLL:  0.18508E-01   163.952

                          DIRECT CALCULATION       COMPUTED FROM RADIATION
     DIFFRACTION (+)    0.6511E+00  0.7421E+00     0.6511E+00  0.7421E+00
     DIFFRACTION (-)    0.1197E+00  0.8949E+00     0.1197E+00  0.8949E+00


     +++++++ ADDED-MASS & DAMPING COEFF. ( NB= 40, K*B/2=  1.0000 ) +++++++

         I  J        ADDED-MASS      DAMPING        ENERGY        ERROR(%)
       ( 1, 1)      0.5831E+00     0.1205E+01     0.1204E+01     0.3517E-01
       ( 2, 2)      0.9761E+00     0.5969E+00     0.5965E+00     0.5948E-01
       ( 3, 3)      0.6037E-03     0.3427E-03     0.3426E-03     0.3604E-01

       ( 1, 2)     -0.1839E-15     0.3053E-15     0.2220E-15
       ( 1, 3)     -0.6261E-02    -0.2032E-01    -0.2031E-01
       ( 2, 1)     -0.5774E-15     0.1730E-15     0.2220E-15
       ( 2, 3)      0.1741E-17    -0.3286E-17    -0.9541E-17
       ( 3, 1)     -0.6258E-02    -0.2032E-01    -0.2031E-01
       ( 3, 2)      0.5150E-18    -0.4337E-18    -0.9541E-17


     +++++ WAVE EXCITING FORCE +++++

               PRESSURE INTEGRAL            HASKIND-NEWMAN
        J      REAL         IMAG          REAL         IMAG           AMP     PHASE(DEG)
        1   0.3035E+00   0.1055E+01    0.3034E+00   0.1055E+01    0.1098E+01    73.952
        2   0.3292E+00   0.6992E+00    0.3290E+00   0.6988E+00    0.7728E+00    64.786
        3  -0.5118E-02  -0.1779E-01   -0.5116E-02  -0.1779E-01    0.1852E-01  -106.048


     +++++ MOTIONS ABOUT 'G'  FOR K*B/2=  1.000 +++++
                  AMP.       PHASE
         SWAY   0.4378E+00    -76.956 (DEG)
         HEAVE  0.9316E+00    -69.197 (DEG)
         ROLL   0.1596E+00    -76.956 (DEG)


     ******* TRANSMISSION & REFLECTION COEFF. ( K*B/2=  1.0000 ) *******
                          CT           CR          CT**2+CR**2
          DIFFRACTION   0.1593E+00   0.9872E+00    0.10000E+01
          MOTION FREE   0.1350E+00   0.9908E+00    0.10000E+01
```